Rational functions: Horizontal and oblique asymptotes NAME:

This worksheet is designed to help us understand why and where horizontal and oblique asymptotes occur on the graphs of rational functions.

We can think of horizontal and oblique asymptotes as the end behavior of rational functions. In other words, horizontal and oblique asymptotes will be determined by what the y values approach at the (left and right) ends of the graph.

Recall a rational function is a function that can be written as a fraction where the top and bottom are both polynomial functions. (Recall that the degree of a polynomial function is the highest exponent of the x's.)

We have seen that the end behavior of a polynomial can be found by looking at the sign (positive or negative) and degree of the leading term.

We will extend this to say that the end behavior (horizontal or oblique asymptotes) of a rational function can be determined by looking at the leading term (and degree) on top and the leading term (and degree) on bottom.

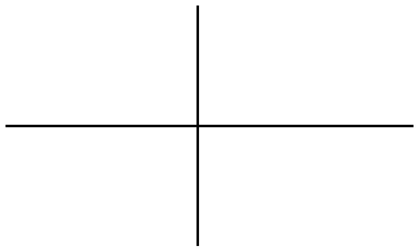
We will see three cases:

1. degree on top **equal to** degree on bottom,
2. degree on top **less than** degree on bottom, and
3. degree on top **greater than** degree on bottom.

Remember, when you graph these functions, put the whole top in parentheses and the whole bottom in parentheses. Also, draw the vertical asymptotes as dashed lines. You should draw in the horizontal or oblique asymptotes as dashed lines as you find them.

Case 1: Degree on top is equal to degree on bottom

1. Graph $y = \dfrac{2x^2 + 3x}{x^2 + 4}$ in the standard window. Quickly copy the graph onto your paper. Remember to put the whole top in parentheses and the whole bottom in parentheses.



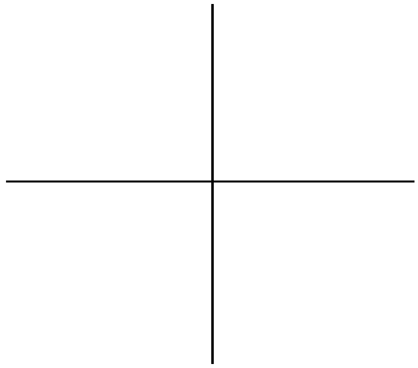
2. Since we are interested in what is happening to the y values as we get really large and really small x values, set your $x$min and $x$max to [-1000, 1000]. (The graph will look very different but that's okay.) Now trace to both ends of the graph. This allows us to look at what is happening to the y values as x gets really large and as x gets really small. What numerical value do the y values appear to be approaching? If you are not sure, change your $x$min and $x$max to [-5000, 5000] and trace out again to the very ends of the graph.

3. How might you find this horizontal asymptote algebraically? Let's investigate.

Recall the top polynomial $y = 2x^2 + 3x$ has the same end behavior as $y = 2x^2$. Likewise, the bottom polynomial has the same end behavior as $y = x^2$. So if we want to know what happens at the end of the graph of $y = \dfrac{2x^2 + 3x}{x^2 + 4}$, we could look at $y = \dfrac{2x^2}{x^2}$. What does this simplify to? Notice you should get the same answer that you got in #2. We will perform this procedure when the degree on top is **equal to** the degree on bottom.

Case 2: Degree on top is less than degree on bottom

4. Consider the function $y = \dfrac{4x^2 - 3x + 6}{5x^3 + 7x - 3}$. Graph it on the standard window and copy it below. Remember to put the whole top in parentheses and the whole bottom in parentheses.

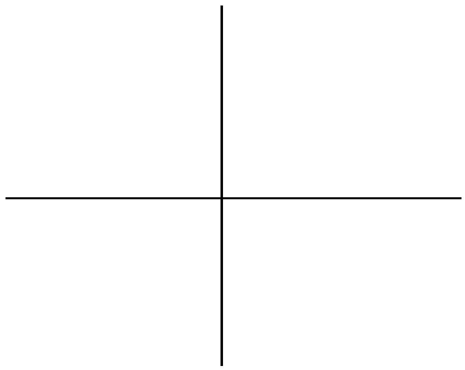


5. Since we are interested in what is happening to the y values as we get really large and really small x values, set your $x$min and $x$max to [-1000, 1000]. (The graph will look very different but that's okay.) Now trace to both ends of the graph. This allows us to look at what is happening to the y values as x gets really large and as x gets really small. What numerical value do the y values appear to be approaching? If you are not sure, change your $x$min and $x$max to [-5000, 5000] and trace out again to the very ends of the graph.

6. It turns out that when the degree on top is **less than** the degree on bottom, the rational function has a horizontal asymptote of $y = 0$. Is that what your graph showed?

Case 3: Degree on top is greater than degree on bottom

7. We'll use the function $y = \dfrac{8x^3 + 4x}{8x^2 - 4x - 7}$ to investigate this case. Graph it below. Use the standard window. Remember to put the whole top in parentheses and the whole bottom in parentheses.



Notice how the ends of the graph do not approach a single value but rather go off at an angle. This indicates an oblique asymptote. You can trace out to the ends and you'll notice the y values just getting bigger and bigger (or more and more negative if you go towards the left end). There is no horizontal asymptote because the y values do not approach a single value.

8. When the degree on top is **greater than** the degree on bottom, the rational function does not have a horizontal asymptote. We use polynomial long division, as described in the notes, to find the equation of the oblique asymptote. Find this oblique asymptote now. (I suggest using spacers for missing terms as I describe with the polynomial division examples in the notes.) Write the asymptote in "*y* = ??" form. Graph the asymptote above with the original function as a dashed line and label it.

9. Without graphing tell the horizontal or oblique asymptotes of the following functions. Write your answers in "y = ??" form. Show the work for the polynomial divisions.

a.) $y = \dfrac{10x^5 - 8x^3 + 6x}{3x^5 + 4x^2 - 7}$

b.) $y = \dfrac{-3x^2 + 8x - 8}{3x + 4}$

c.) $y = \dfrac{4.5x^2 - 3x}{2x^2 + 7x - 9}$

d.) $y = \dfrac{-7x^5 + 4x^4 - 7x^3 + 9}{8x^8 - 9x^2 + 4}$

e.) $y = \dfrac{x^3 - x}{2x^3 - 3x}$

f.) $y = \dfrac{14x + 2}{2x - 3}$

g.) $y = \dfrac{6x^3 + 4}{15x^5 - 9x + 3}$

h.) $y = \dfrac{-4x^4 + 7x^2 - 4}{8x^2 - 6}$