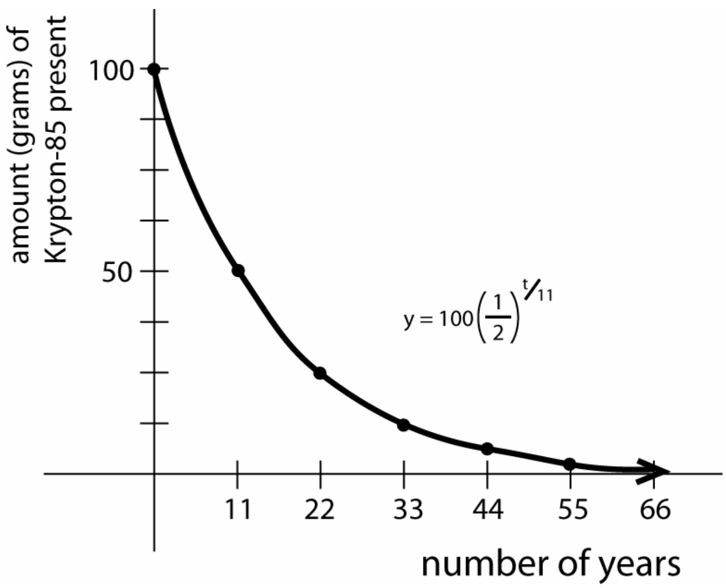
Logarithmic and exponential applications Solutions NAME:
Exponential decay and growth

1. The half-life of Krypton-85 is 11 years. Suppose we have 100 grams of Krypton-85 now. Complete the table for the amount of Krypton-85 after t years. (Notice the values of t are given in increments of 11 years.)

number of years	$\boldsymbol{A(t)}$ **= amount present after** $\boldsymbol{t}$ **years**
0	100
11	$50 = 100\left(\frac{1}{2}\right)$
22	$25 = 100\left(\frac{1}{2}\right)^2$
33	$12.5 = 100\left(\frac{1}{2}\right)^3$
44	$6.25 = 100\left(\frac{1}{2}\right)^4$
⋮	
t	$100\left(\frac{1}{2}\right)^{t/11}$

Now graph the relationship between years and amount of Krypton-85 present by plotting the points in the table. Notice the graceful curve of the exponential relationship. Because the decline is not constant, the graph has a curved shape, instead of a linear look.



In the table, the 50 was gotten by multiplying 100 times ½. Multiplying by another ½ gets us the next entry. In fact, each entry is the last times ½. If we write each as 100 times ½ raised to some power, we see a connection between the number of years and the exponent. Thinking through it this way gets us $y = 100\left(\frac{1}{2}\right)^{t/11}$.

2. Iodine-131 decays according to the function $A(t) = A_0 e^{-.087t}$ where A_0 is the initial amount present and $A(t)$ is the amount left after t days. On January 1st, there were 30 grams of it in my basement. How long will it take for there to be only .65 grams of it?

$$A(t) = A_0 e^{-.087t}$$
$$.65 = 30e^{-.087t}$$

Remember, start out with the original equation and make sure you know what the variables represent. Put 30 in for A_0 because that's the initial amount. This gives you $A(t) = 30e^{-.087t}$.

Now we want to know how long (or t) it takes for the amount to be .65, which is denoted by $A(t)$. So substitute that to get $.65 = 30e^{-.087t}$. Now solve for t.

$$.65 = 30e^{-.087t}$$
$$.0217 = e^{-.087t}$$
$$\ln(.0217) = \ln(e^{-.087t})$$
$$-3.8320 = -.087t$$
$$44.05 = t$$

Isolate the exponential factor $e^{-.087t}$ *and then take the natural log of both sides. This unburies the variable from the exponent position.*

To simplify the right (third line), we think through
$\ln(e^{-.087t}) = -.087t * \ln(e) = -.087t * 1 = -.087t$.

Round intermediate answers to at least four decimal places. Did you get 44 days as the final answer? So on February 14th, there will be only .65 grams of Iodine-131 in my basement.

Rounding your intermediate answers to four decimal places will help your final answer be more accurate. However, it's even better if you can get your calculator to use the exact values.

3. The world's population is an example of exponential growth. According to data published by the United Nations, the world population in 1975 was approximately 4 billion. The formula for the world's population can be given as $N(t) = 4e^{.02t}$ where $N(t)$ is the population (in billions) t years after 1975. When will the population reach 8 billion? Round intermediate answers to four decimal places and your final answer to the nearest whole number.

$$N(t) = 4e^{.02t}$$
$$8 = 4e^{.02t}$$

Make sure you know what the variables represent. You want to know the t value that makes $N(t)$ equal to 8. So solve the equation $8 = 4e^{.02t}$. Remember this involves isolating the $e^{.02t}$ part and then taking the natural log of both sides to unbury the t from the exponent.

$$8 = 4e^{.02t}$$
$$2 = e^{.02t}$$
$$\log_e 2 = .02t$$
$$.6931 = .02t$$
$$34.66 = t$$

Here, we again isolate the exponential part $e^{.02t}$ *before anything else. I then rewrote the exponential equation in logarithmic form. I have to keep in mind the equivalent relationship between* $x = 2^y$ *and* $y = \log_2 x$. *Remember, log, base e, is the natural log. So I used the LN button on my calculator to find* $\log_e 2$.

Did you get 35 years? So in the year 2010, the world's population should be 8 billion.

4. The half-life of radioactive potassium is 1.3 billion years. If 10 grams is present now, how much is present after 1000 years?

Use the formula $A(t) = A_0\left(\frac{1}{2}\right)^{t/h}$ where A_0 is the initial amount, h is the half-life, and $A(t)$ is the amount present after t years. Notice here we are given t so we will not need to solve like those we have seen. For convenience of calculations, notice 1.3 billion is $1.3\text{x}10^9$. Round your answer to 10 or so decimal places. Notice how very little of the substance is gone after 1000 years. That's what makes radioactive material such a huge problem!

$$A(t) = 10\left(\frac{1}{2}\right)^{1000/1.3\times10^9} = 9.999994668$$

Use the EE button for scientific notation. It is the key directly above the 7 key. Make sure you get parentheses around the entire exponent.

Try some on your own:

1. We saw from the discussion of Krypton-85 that the formula that gives us the amount present after t years is $A(t) = A_0\left(\frac{1}{2}\right)^{t/11}$. Assume there are 100 grams of it now. How long will it take so that there is only 1 gram left?

$$1 = 100\left(\frac{1}{2}\right)^{t/11}$$

$$.01 = \left(\frac{1}{2}\right)^{t/11}$$

$$\ln(.01) = \ln\left(\frac{1}{2}\right)^{t/11}$$

$$-4.6052 = t/11 * \ln\left(\frac{1}{2}\right)$$

$$-4.6052 = t/11 * (-.6931)$$

$$73.08 = t$$

Divide by the 100 to isolate the exponential part. Then take the natural log of both sides. The second to last line has t being divided by 11 and multiplied by -.6931. So, to undo that and get t alone, divide by -.6931 and multiply by 11. It will take 73 years for our 100 grams of Krypton-85 to degrade to 1 gram.

2. Four bacteria are in a Petri dish. Every 5 seconds, the number of bacteria doubles. Use the table to develop a formula for the number of bacteria after t seconds. Then answer the question, “After how many seconds will the number of bacteria reach 1000?”

***t* = number of seconds**	**N(t) = number of bacteria**
0	4
5	$8 = 4 * 2$
10	$16 = 4 * 2^2$
15	$32 = 4 * 2^3$
20	$64 = 4 * 2^4$
⋮	
t	$4 * 2^{t/5}$

$$N(t) = 4 * 2^{t/5}$$

$$1000 = 4 * 2^{t/5}$$

$$250 = 2^{t/5}$$

$$\ln(250) = \ln\left(2^{t/5}\right)$$

$$5.5215 = t/5 * \ln(2)$$

$$5.5215 = t/5 * (.6931)$$

$$39.83 = t$$

3. Radium has a half-life of 1690 years. If you have 200 grams of it today, how long will it be until you have 50 grams? (Think about this one. You may not need to do complicated calculations.)

Years	Amount
0	*200*
1690	*100*
3380	*50*

Just make a table of time versus amount. Since you only know what happens every 1690 years, write the time as 0, 1690, 3380, etc. Notice you do not have to get far to see that if we start with 200 grams, we'll have 50 grams in 3380 years.

4. Radium has a half-life of 1690 years. Suppose you just got a new shipment of 5 grams of the radioactive substance today. How long will it take for your sample to decay to only .5 grams?

$$N(t) = N_0\left(\frac{1}{2}\right)^{t/1690}$$

$$.5 = 5\left(\frac{1}{2}\right)^{t/1690}$$

$$.1 = \left(\frac{1}{2}\right)^{t/1690}$$

$$\ln(.1) = \ln\left(\frac{1}{2}\right)^{t/1690}$$

$$-2.3026 = t/1690 * \ln\left(\frac{1}{2}\right)$$

$$-2.3026 = t/1690 * -.6931$$

$$5614.06 = t$$

Use the half-life formula. Isolate the exponential part. Then take the natural log of both sides. Solve like before. The radium will decay to .5 grams in 5614 years.

5. A certain insect population grows exponentially. The size $P(t)$ of the population after t days obeys the function $P(t) = 650e^{.035t}$. When will the population reach 1600 insects?

$$1600 = 650e^{.035t}$$

$$2.4615 = e^{.035t}$$

$$\ln(2.4615) = \ln e^{.035t}$$

$$.9008 = .035t$$

$$25.74 = t$$

Again, isolate the exponential part. Then take the natural log of both sides. Again, notice how this unburies the variable from the exponent position. The insect population will reach 1600 in a little under 26 days.