Exploration: Elimination of Possibilities Technique NAME:
Chapter 3, Number 10 “The Three Squares”

Problem: Three cousins, Bob, Chris, and Phyllis, are watching TV together. They got to talking about their ages. Bob (the oldest) remarks that their ages were all between 11 and 30. Phyllis remarked that the sum of their ages was 70. Chris, the youngest, excitedly called out, “If you write out the squares of our ages, all of the digits 1 through 9 appear exactly once.” How old was each cousin?

Guided Solution: At first glance, this seems rather daunting. So many possibilities! (In fact, there are 1,140 possible triplets among the numbers 11 through 30.) We will attempt to eliminate possibilities to narrow in on our answer. First, let’s consider Bob’s clue that all three ages are between 11 and 30. Here, I have written those numbers down along with their squares. Using Chris’ clue, which individual ages can be ruled out?

Possible Age	**Square of Age**		**Possible Age**	**Square of Age**
11	121		21	441
12	144		22	484
13	169		23	529
14	196		24	576
15	225		25	625
16	256		26	676
17	289		27	729
18	324		28	784
19	361		29	841
20	400		30	900

Think only of Chris’ clue. How can we eliminate some of these ages right away? Cross them off the list and continue.

Write out the list of ages that we now know to be possible. Admire how we have whittled down the original list considerably.

We could certainly continue by looking at every triplet (three of the possible ages) and see if they sum to 70, but there are 220 such possible triplets. So, let's attempt to whittle down the field a little more.

Thinking about groups of three ages is hard. So let's make our job a little easier. What if we consider just two ages at a time to see if they could possibly be two of the three numbers in our triplet? If we find a pair that could *not* work together because of Chris' clue, then we can say the triplet could *not* include those two ages together. That could further eliminate many possibilities. Consider the table below to help this along.

Along the top and left column, I have written the ages that are left after the elimination from page 1. I have crossed out all pairings along the diagonal since two cousins could not be the same age because of Chris' clue. The squares of the ages are written down the rightmost column for easy reference.

I went down the column for "13" and crossed out all other ages that could *not* possibly be in a triplet with 13 because of Chris' clue. I then copied these marks along the row for 13. Do you see why we can do this? Do the same for the other possible ages.

Of course, alternative logic can be used to eliminate pairs of numbers. For instance, can you think of another reason that 14 and 16 *cannot* be two of the three ages?

	13	14	16	17	18	19	23	24	25	27	28	29	
13	X	X	X	X		X	X	X	X	X		X	$13^2 = 169$
14	X	X											$14^2 = 196$
16	X		X										$16^2 = 256$
17	X			X									$17^2 = 289$
18					X								$18^2 = 324$
19	X					X							$19^2 = 361$
23	X						X						$23^2 = 529$
24	X							X					$24^2 = 576$
25	X								X				$25^2 = 625$
27	X									X			$27^2 = 729$
28											X		$28^2 = 784$
29	X											X	$29^2 = 841$

Here, I have completed this table below.

	13	14	16	17	18	19	23	24	25	27	28	29	
13	X	X	X	X		X	X	X	X	X		X	$13^2 = 169$
14	X	X	X	X		X	X	X	X	X		X	$14^2 = 196$
16	X	X	X	X	X	X	X	X	X	X			$16^2 = 256$
17	X	X	X	X	X		X		X	X	X	X	$17^2 = 289$
18			X	X	X	X	X		X	X	X	X	$18^2 = 324$
19	X	X	X		X	X		X	X			X	$19^2 = 361$
23	X	X	X	X	X		X	X	X	X			$23^2 = 529$
24	X	X	X			X	X	X	X	X	X		$24^2 = 576$
25	X	X	X	X	X	X	X	X	X	X			$25^2 = 625$
27	X	X	X	X	X		X	X	X	X	X		$27^2 = 729$
28				X	X			X		X	X	X	$28^2 = 784$
29	X	X		X	X	X					X	X	$29^2 = 841$

This does help us immensely but we are still not done. Starting at the left side, we see the only two numbers that could possibly work with 13 (again, considering Chris' clue) are 18 and 28. However, this triplet would *not* work? Do you know why? (I can see two reasons, give one.)

In fact, the same is true for the triplet 14, 18, and 28. What other possible triplets do you see from the table? We will form a table and explore them one by one. This may seem difficult but if we compare this chore with that of trying out 220 (or, egad! 1140) possibilities, we see we have improved our situation greatly by the process of elimination.

So, next you will see an organized list of possible triplets derived from the table above. Do you see how I made the list?

Possible triplet from above table	Reason it cannot work
13, 18, 28	sum is not a multiple of 10; and 18^2 and 28^2 share digits
14, 18, 28	18^2 and 28^2 share digits
16, 28, 29	sum is not a multiple of 10
17, 19, 24	
18, 13, 14	sum is not a multiple of 10
18, 13, 24	sum is not a multiple of 10
18, 14, 24	sum is not a multiple of 10
19, 17, 23	sum is not a multiple of 10
19, 17, 27	sum is not a multiple of 10
19, 17, 28	sum is not a multiple of 10
19, 23, 27	sum is not a multiple of 10
19, 23, 28	
19, 27, 28	sum is not a multiple of 10
23, 19, 29	sum is not a multiple of 10
23, 28, 29	
24, 17, 18	sum is not a multiple of 10
24, 17, 29	
24, 18, 29	sum is not a multiple of 10
25, 28, 29	sum is not a multiple of 10
27, 19, 29	sum is not a multiple of 10
28, 13, 14	sum is not a multiple of 10
28, 13, 16	sum is not a multiple of 10
28, 13, 19	
28, 13, 23	sum is not a multiple of 10
28, 13, 25	sum is not a multiple of 10
28, 14, 16	sum is not a multiple of 10
28, 14, 19	sum is not a multiple of 10; and 14^2 and 19^2 share digits
28, 14, 23	sum is not a multiple of 10
28, 14, 25	sum is not a multiple of 10
28, 16, 19	sum is not a multiple of 10
28, 16, 23	sum is not a multiple of 10
28, 16, 25	sum is not a multiple of 10
28, 19, 25	sum is not a multiple of 10
28, 23, 25	sum is not a multiple of 10
29, 16, 23	sum is not a multiple of 10
29, 16, 24	sum is not a multiple of 10
29, 16, 25	
29, 16, 26	sum is not a multiple of 10
29, 23, 24	sum is not a multiple of 10
29, 23, 25	sum is not a multiple of 10
29, 23, 26	sum is not a multiple of 10
29, 24, 25	sum is not a multiple of 10
29, 24, 27	sum is not a multiple of 10
29, 25, 27	sum is not a multiple of 10

This seems like a ridiculous list. But we have narrowed our search to 44 triplets.

Add *just* the ones digits of these numbers to eliminate more quickly. Why will that work?

There is often more than one reason to eliminate a triplet.

So, once we eliminate the vast majority of these 44 possibilities because we see they could not possibly add to 70, we get a much shorter list. (Remember, we did *not* have to actually add the numbers, just the ones digits.) We could go through these one by one, investigating each to see if the triplet has a sum of 70 and the digits 1-9 appear exactly once in their squares. Do this now.

Possible triplet from above table	**Is the sum 70?**	**What are the squares of each number? Do the digits 1-9 appear exactly once?**
17, 19, 24		
19, 23, 28		
23, 28, 29		
24, 17, 29		
28, 13, 19		
29, 16, 25		

I repeat this table from page 1 to make the above job easier.

Possible Age	**Square of Age**		**Possible Age**	**Square of Age**
11	121		21	441
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16	256		26	676
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So, what are the cousins' ages?

Bob is ___________.

Phyllis is _________.

Chris is __________.