What is the square root of a negative number?

College algebra
Class notes
The Complex Numbers and Quadratic Equations (section 1.4)

When we evaluate $\sqrt{25}$, we ask ourselves "What (non-negative) number squared makes 25?"

$\sqrt{25} = 5$ because $5^2 = 25$

But what about $\sqrt{-25}$? What number squared makes -25 ?

It does not exist in the real numbers

$(?)^2 = -25$

It turns out that *no* real number can be squared to make -25. But instead of leaving problems with square roots of negative numbers unresolved, we made up complex or imaginary numbers. We start off by defining the basic building block of complex numbers.

We will let $i = \sqrt{-1}$.

So, now we can simplify $\sqrt{-25}$ as $\sqrt{25}\sqrt{-1}$ or $5i$.

By the way, what do you think i^2 is equal to? This fact will be useful to us when we manipulate complex numbers.

$i^2 = \sqrt{-1}\sqrt{-1}$

$\boxed{i^2 = -1}$

If $i = \sqrt{-1}$, then $i^2 = \sqrt{-1}\sqrt{-1}$?

If $\sqrt{9}\sqrt{9} = 9$, then what is $\sqrt{-1}\sqrt{-1}$?

How would you verify that $5i$ is the number that you square to get -25? Do it now.

$(5i)^2 = 5i \cdot 5i = 25i^2 = 25(-1) = -25$ ✓

Hence $(5i)^2 = -25$ and so $\sqrt{-25} = 5i$.

expl 1: Write the following in terms of i.

a.) $\sqrt{-36}$

$= \sqrt{36}\sqrt{-1}$

$= 6i$

b.) $\sqrt{-32}$

$= \sqrt{32}\sqrt{-1}$

$= \sqrt{32} \cdot i$

$= 4\sqrt{2} \cdot i$

$= 4i\sqrt{2}$

$32 = 2 \cdot 16$

Side note: Factors versus terms:

This concept will follow you through algebra and further.

terms: things we are adding (or subtracting)
expls: $x + 4$ or $2x + 3$ or $4x^2 + 3x - 6$

factors: things we are multiplying
expls: $5 \cdot x$ or $3(x + 2)$ or $4 \cdot x^2$

Could be thought of as $4 \cdot x \cdot x$ or $2 \cdot 2 \cdot x \cdot x$. What are the factors then?

The reason this distinction is important is because we will use these words a lot. When we simplify expressions, what we do and why (the rules that govern real numbers) depends a lot on if we are adding (or subtracting) versus if we are multiplying (or dividing).

Can you make up your own example of terms and factors?

$3x + 5$ versus $3x \cdot 5$
terms (added) factors (multiplied)

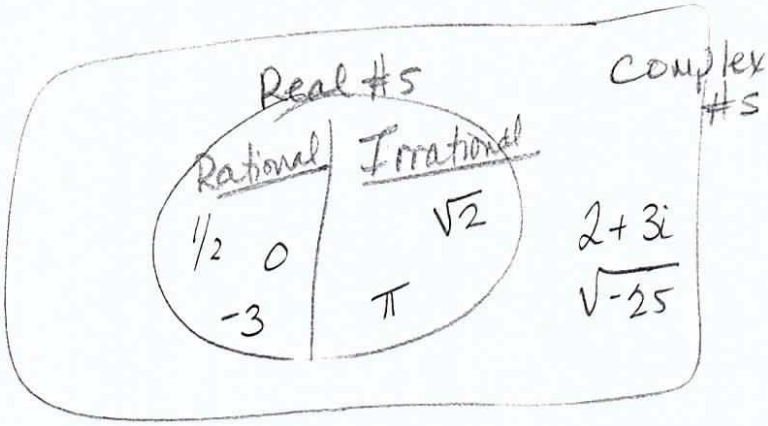
Definition: Complex number: A complex number is a number that could be written in the form $a + bi$ where a and b are real numbers. The a is called the **real part** and the b is called the **imaginary part**.

expls: $0 + 5i$ $\quad 2 + 4i$ $\quad -6 - 7i$ $\quad 14$ $\quad \sqrt{5} + 4i$

a : real part
b : imaginary part

Can you see how each could be written in the form $a + bi$? What are a and b in each case?

Do you know the relationship between real numbers and complex numbers?



Complex, non-real numbers are called "imaginary".

Definition: Conjugate of a complex number: The conjugate of the complex number $a + bi$ is said to be $a - bi$. These two numbers are called complex conjugates. This is sometimes written as $\overline{a+bi} = a - bi$. (The conjugate of a complex number z is written $\bar{z}$.)

Simply change the sign of the i term to form the conjugate.

$\overline{14} = 14$

expl 2: Write the conjugates of each complex number below.

Complex number	$5i$ $0+5i$	$2+4i$	$-6-7i$	14 $14+0i$	$\sqrt{5}+4i$
Its conjugate	$-5i$	$2-4i$	$-6+7i$	14	$\sqrt{5}-4i$

OR $0-5i$ OR $14-0i$

The product of a complex number and its conjugate:

What is the product of a complex number and its conjugate? Find out by FOILing out $(a + bi)(a - bi)$. Is the product complex (but *not* real) or just a plain, old real number? How can you tell?

$$(a+bi)(a-bi) = a^2 + \cancel{abi} - \cancel{abi} - b^2i^2$$
$$= a^2 - b^2i^2$$
$$= a^2 - b^2(-1)$$
$$= a^2 + b^2$$

Plain, old real number (no i present

We will use this fact when we divide complex numbers. But first, let's learn to add, subtract, and multiply them.

What would you do if it was $(2 + 4x) + (6 - 3x)$?

expl 3: Add or subtract. Write the result in the form $a + bi$.

a.) $(2 + 4i) + (6 - 3i)$

$= 8 + i$

b.) $(-5 + 7i) + (-9i)$

$= -5 - 2i$

c.) $(2 + 4i) - (6 - 3i)$

$= 2 + 4i - 6 + 3i$

$= -4 + 7i$

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expl 4: Multiply. Write the result in the form $a + bi$.

a.) $-5i \cdot 6i$ $= -30i^2$

$= -30(-1)$

$= 30$

What would you do if it was $(4 + 3x)(5 - 4x)$? FOIL

b.) $(4 + 3i)(5 - 4i)$ $= 20 + 15i - 16i - 12i^2$

$= 20 - i - 12(-1)$

$= 20 - i + 12 = 32 - i$

What was i^2 again? $i^2 = -1$

Can you do these on the calculator? yes

c.) $\left(\sqrt{5} + 4i\right)\left(\sqrt{5} - 4i\right)$

$= 5 + \cancel{4i\sqrt{5}} - \cancel{4i\sqrt{5}} - 16i^2$

$= 5 - 16i^2 = 5 - 16(-1) = 5 + 16 = 21$

Dividing complex numbers involves eliminating the i from the bottom of the fraction. Recall that multiplying a complex number by its conjugate does just that. We will have to simplify a bit more to get the final answer in the form $a + bi$.

expl 5: Divide. Write the result in the form $a + bi$.

$$\frac{5}{(2 + 3i)} \cdot \frac{(2-3i)}{(2-3i)} = \frac{5(2-3i)}{4+9}$$

$$= \frac{5(2-3i)}{13}$$

$$= \frac{10-15i}{13} = \boxed{\frac{10}{13} - \frac{15}{13}i}$$

What is the conjugate of $2 + 3i$? $2-3i$

expl 6: Divide. Write the result in the form $a + bi$.

$$\frac{(2 + 5i)}{(3 + 3i)} \cdot \frac{(3-3i)}{(3-3i)}$$

$$= \frac{6+15i-6i-15i^2}{9+9}$$

$$= \frac{6+9i-15(-1)}{18} = \frac{6+9i+15}{18}$$

$$= \frac{21+9i}{18} = \frac{21}{18} + \frac{9i}{18} = \boxed{\frac{7}{6} + \frac{1i}{2}}$$

What is the conjugate of $3 + 3i$? $3-3i$

Simplify, simplify, simplify...

Finding the reciprocal of a complex number:

If asked to find the reciprocal of a complex number $a + bi$, we write it as $\dfrac{1}{a+bi}$ and then multiply both top and bottom by its conjugate. Write your final answer in the form $a + bi$.

Powers of i : So $i^1 = i$ and $i^2 = -1$, but what are the higher powers of i equal to? Fill in the following powers of i to elicit a pattern.

Recall $i^3 = i^2 \cdot i$,
$i^4 = i^2 \cdot i^2$,
$i^5 = i^2 \cdot i^3$,
etc.

	i^{-3}	$i^1 = i$	$i^5 = i$	$i^9 = i$	$i^{13} = i$	i^{21}
i^{-6}	i^{-2}	$i^2 = -1$	$i^6 = -1$	$i^{10} = -1$	$i^{14} = -1$	
i^{-5}	i^{-1}	$i^3 = -i$	$i^7 = -i$	$i^{11} = -i$	$i^{15} = -i$	
i^{-4}	$i^0 = 1$	$\boxed{i^4 = 1}$	$\boxed{i^8 = 1}$	$\boxed{i^{12} = 1}$	$\boxed{i^{16} = 1}$	$\boxed{i^{20}}$

$i^4 = i^2 \cdot i^2$
$= (-1)(-1) = 1$

$i^5 = i^4 \cdot i$
$= 1 \cdot i$
$= i$

expl 7: Use the pattern above to find each of the following powers of i.

Where would i^{21} fall in the table above?

a.) $i^{21} = i$ (Row 1)

b.) $i^{84} = 1$ (Row4)

Continue the pattern above to the left side for negative (and zero) powers.

c.) $i^{-6} = -1$

Worksheet: Manipulating complex numbers 3

We will practice adding, subtracting, multiplying, and dividing complex numbers. We will also explore the powers of i.

Solving Quadratic Equations:

You might recall the quadratic formula is a common way to solve quadratic equations. We will see complex numbers pop up in these problems. Let's start with the basics and we will see how complex numbers play a role.

Recall: Definition: Quadratic equation: A quadratic equation is an equation that could be written in the form $ax^2 + bx + c = 0$ where a is *not* zero.

examples:
- $5x^2 + 2x + 16 = 0$
- $(x + 5)(x - 8) = 0$
- $4x^2 + 48x = 12$
- $-5x^2 + 4x - 21 = 0$

What kind of equation does it become when a is zero?

They may not *look* to be in the form $ax^2 + bx + c = 0$. Are they really quadratic equations?

counterexamples: $\frac{x-3}{x+9} = \frac{6}{x+2}$, $\sqrt{4x+5} = 7$, $2|5x - 3| + 7 = 21$, $5x + 8 = 0$

We have other methods to solve these, non-quadratic equations, don't we?

Solving Equations by the Quadratic Formula:

Some old guy long ago (actually many old guys independently throughout time), solved the general quadratic equation $ax^2 + bx + c = 0$ (where a is *not* zero) and found the solutions to be $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. We will piggyback on their work to solve our own equations.

expl 8: Solve to find the exact solution(s).

$x^2 + x + 7 = 0$

What are a, b, and c?

What happens under the radical? Do you see how this makes complex solutions?

negative under radical!

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1^2 - 4(1)(7)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{1-28}}{2} = \frac{-1 \pm \sqrt{-27}}{2} = \frac{-1 \pm 3i\sqrt{3}}{2}$$

Definition: Discriminant: In the quadratic formula, the expression under the radical, $b^2 - 4ac$, is called the **discriminant**. It can be used to glean information about an equation's solutions *without* fully solving it.

Our first question is, "How many real solutions does an equation have if the discriminant is **negative**?" This happened in the previous example.

no real solns (2 complex solns)

What about when the discriminant is **positive**? How many real solutions would those equations have? What is the nature of the solutions?

2 real solns (not complex)

What about when the discriminant is **zero**? What happens to the solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $\to 0$

when $b^2 - 4ac$ is 0? Are the solution(s) real or imaginary (complex but *not* real)? How many solutions are there?

1 real soln.

$x = \dfrac{-b \pm \sqrt{0}}{2a} = \dfrac{-b}{2a}$

Use this table to summarize the information.

If the discriminant is …	then there will be ______________ solutions.
negative,	2 complex
positive,	2 real (not complex)
zero,	1 real (not complex)

expl 9: Use the discriminant to determine the character of the solutions to the equation below. In other words, determine if there is one real (repeated) solution, two (unequal) real solutions, or two complex solutions that are conjugates of each other.

$3 - 6x^2 = -5x + 1$

$+5x \quad -1 \qquad +5x \quad -1$

What are a, b, and c?
($ax^2 + bx + c = 0$)
Be careful!

Only calculate $b^2 - 4ac$.
Do *not* use the whole
quadratic formula.

$-6x^2 + 5x + 2 = 0$

$a = -6$
$b = 5$
$c = 2$

$b^2 - 4ac$

$= 5^2 - 4(-6)(2)$

$= 73$ → So eqn $3 - 6x^2 = -5x + 1$ has 2 real (not complex) solns.