We will talk about symmetry, when a function is going up or down, extreme values, and how fast a function is changing.

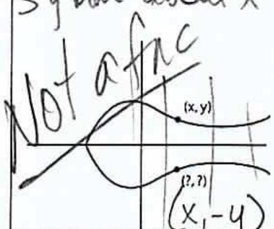
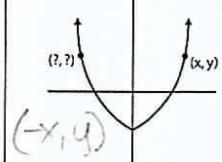
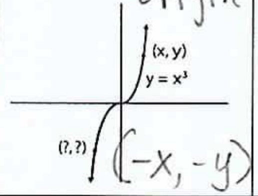
College Algebra
Class Notes
Properties of Functions (section 3.3)

We will revisit the concept of symmetry and explore concepts such as the largest or smallest y-value on a graph. The concepts are intuitive but we will use algebra to describe them.

due Mon, Sep. 25

Worksheet: Investigating functions 3:
We work on the definition of a function, domain, and finding function values graphically and algebraically.

Recall: Symmetry: Do you remember which graph below is symmetric about the y-axis? Which is symmetric about the x-axis? Which is symmetric about the origin?

Symm about x Not a fnc  (x, -y)	symm about y  (-x, y)	Symm about origin  (-x, -y)	Do you remember the coordinates of the unknown points?

Recall: Algebraic tests for symmetry:
The pictures above help justify the following tests.

To test a relationship for symmetry about the …

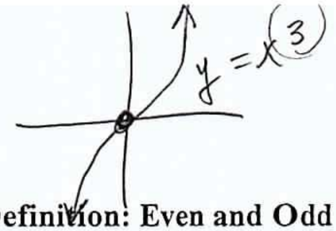
~~**x-axis:** Replace y with $-y$. If the equation is equivalent, then the relationship is symmetric with respect to the x-axis.~~

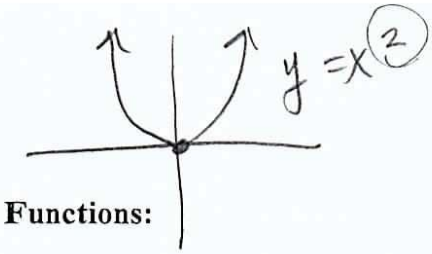
y-axis: Replace x with $-x$. If the equation is equivalent, then the relationship is symmetric with respect to the y-axis.

origin: Replace x with $-x$ and replace y with $-y$. If the equation is equivalent, then the relationship is symmetric with respect to the origin.

We are focusing on functions. Which above graph and symmetry could *not* be of a function? Cross it off because we will think of it no more. No more, I say!

We have the following definitions.





Definition: Even and Odd Functions:

If a function is symmetric about the y-axis, we call it **even**.

If a function is symmetric about the origin, we call it **odd**.

Picture $y = x^2$ and $y = x^3$. Draw them here.

We can frame the earlier algebraic test in terms of function notation.

A function is **even**, if and only if for every x in the domain, $\boxed{\text{we know } f(-x) = f(x).}$

A function is **odd**, if and only if for every x in the domain, $\boxed{\text{we know } f(-x) = -f(x).}$

We'll use this to check whether a function is even, odd, or neither.

What does "if and only if" mean?

Could a function be both even and odd?

? ? ?

expl 1: For the following functions, test if it is even, odd, or neither.

Find $f(-x)$. Is it equal to $f(x)$ or $-f(x)$ or neither?

a.) $f(x) = x^3 + x$

$$f(-x) = (-x)^3 + (-x)$$
$$= -x^3 - x$$
$$= -(x^3 + x)$$
$$= -f(x)$$

So, $f(x) = x^3 + x$ is odd.

$$(-x)^3 = (-x)(-x)(-x)$$
$$= -1 \cdot -1 \cdot -1 \cdot x \cdot x \cdot x$$
$$= -x^3$$

b.) $f(x) = 3x^2 - 5x^4$

$$f(-x) = 3(-x)^2 - 5(-x)^4$$
$$= 3x^2 - 5x^4$$
$$= f(x)$$

So, $f(x) = 3x^2 - 5x^4$ is even.

$$(-x)^2 = (-x)(-x) = x^2$$
$$(-x)^4 = (-x)(-x)(-x)(-x) = x^4$$

c.) $f(x) = x^2 + 3x - 4$

$$f(-x) = (-x)^2 + 3(-x) - 4$$
$$= x^2 - 3x - 4$$

This is neither $f(x)$ nor $-f(x)$,

So $f(x) = x^2 + 3x - 4$ is neither even nor odd.

Here's an optional problem to stretch your mind.

~~expl 2: Prove that the product of an odd function and an even function will always be odd.~~

Skip

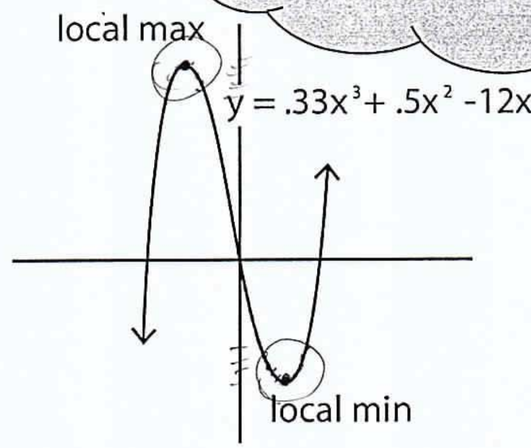
Let $f(x)$ be an even function and $g(x)$ be an odd function.

We need to show
$$(f \cdot g)(-x) = -(f \cdot g)(x)$$
OR
$$f(-x) \cdot g(-x) = -f(x) \cdot g(x)$$

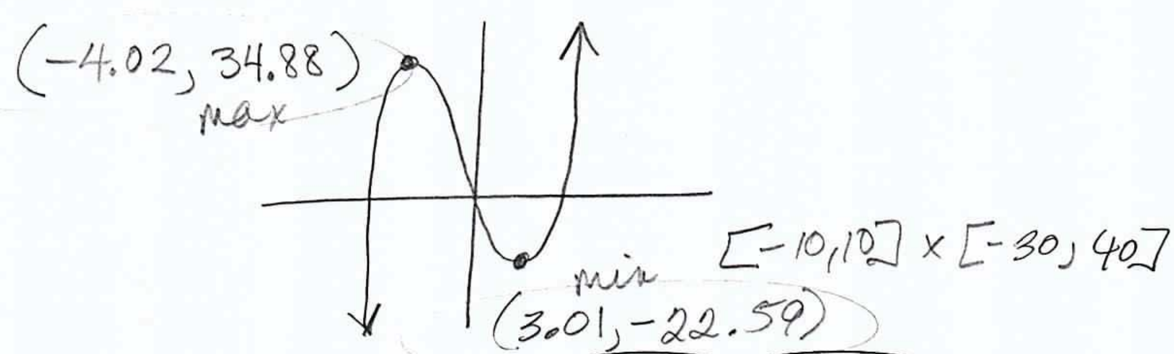
Definition: Local (or relative) extrema:

A **local minimum** is the point (technically, the y-value) on the graph where the **y-value is the smallest**, in that area of the graph.

A **local maximum** is the point (technically, the y-value) on the graph where the **y-value is the largest**, in that area of the graph.



expl 3: Use your calculator to find the local maximum and minimum of the function pictured above. Do *not* just TRACE but rather use the Maximum and Minimum calculator functions.



Use the Minimum (or Maximum) function (under the CALC menu on the TI-83 or 84) to find the smallest (or largest) y-value and the x-value that makes it.

For reference

Worksheet: Finding maximums and minimums on your graphing calculator (82, 83, 85, 86):

This worksheet shows how to find the points of maximum or minimum y-values on your graphing calculator. Instructions for the TI83 will work for TI84's.

Local (Relative) Extrema: A more precise way to state this definition:

Suppose f is a function for which $f(c)$ exists for some c in the domain of f. Then

> $f(c)$ is a **local (or relative) minimum** if there exists an open interval I containing c such that $f(c) \le f(x)$ for all x in I; or
> $f(c)$ is a **local (or relative) maximum** if there exists an open interval I containing c such that $f(c) \ge f(x)$ for all x in I.

Absolute Extrema (Minimums and Maximums):

The concept of local maxes and mins focuses on a small interval around a given point. Consider the definition here and how it differs.

Definition: Absolute Maximums and Minimums:

Let f be some function defined on the interval I (meaning I is the whole domain of f).

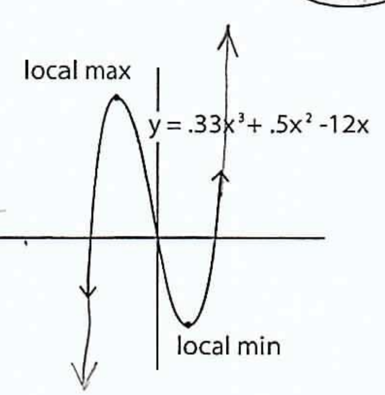
> If there is a number u in I for which $f(u) \ge f(x)$ for all x in I, then f has an absolute maximum at u, and the number $f(u)$ is the **absolute maximum** of f on I.
> If there is a number v in I for which $f(v) \le f(x)$ for all x in I, then f has an absolute minimum at v, and the number $f(v)$ is the **absolute minimum** of f on I.

This max or min has to the smallest or largest on the *whole* graph, *not* just an interval surrounding it.

Again, it is the y-value that is said to be the max or min.

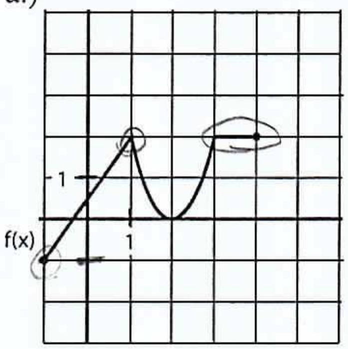
Think back to this graph. What are the absolute maximum and minimum? Do they even exist?

They do not exist



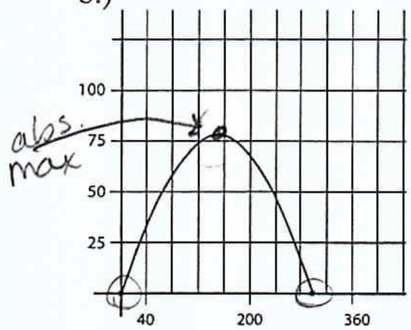
expl 4: For each function below, determine the absolute maximum and absolute minimum if they exist. If an extremum does *not* exist, explain why.

a.)



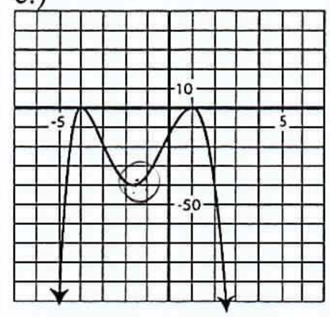
abs min : $y = -1$
abs max : $y = 2$

b.)



abs min: $y = 0$
abs max: $y = 77$

c.)



abs min: none
abs max: $y = 0$

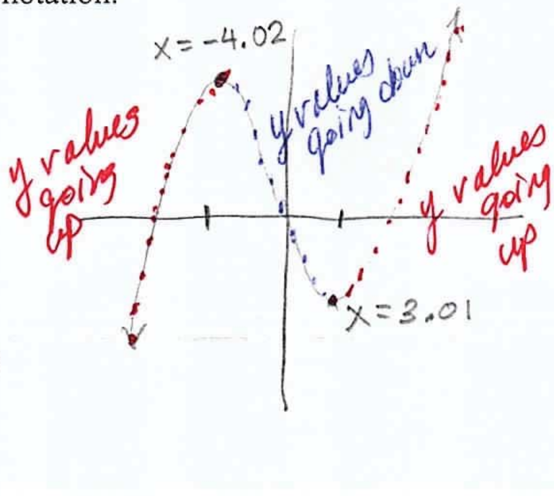
By the way, for part *b*, the endpoints' *y*-values of 0 are *not* considered to be a *local* min because there is *not* an *open* interval containing it. This does *not* come up when finding the absolute minimum.

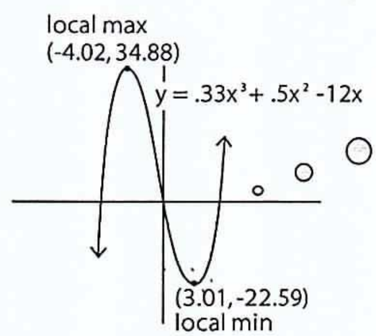
Increasing and decreasing functions:

We will investigate where the graph's *y*-values are increasing and where they are decreasing.

We look at **what is happening to the *y*-values as we go left to right** on the graph. And then we **write the intervals of *x*-values** that result in those increasing or decreasing parts of the graph. We will usually use interval notation.

expl 5: Where is this function increasing and decreasing? Write your answers in interval notation.





Think left to right.

inc: $(-\infty, -4.02], [3.01, \infty)$

dec: $[-4.02, 3.01]$

A more precise way to state this definition:
A function is **increasing** on an interval I, if for any choice of x_1 and x_2 in that interval where $x_1 < x_2$, then $f(x_1) < f(x_2)$.

A function is **decreasing** on an interval I, if for any choice of x_1 and x_2 in that interval where $x_1 < x_2$, then $f(x_1) > f(x_2)$.

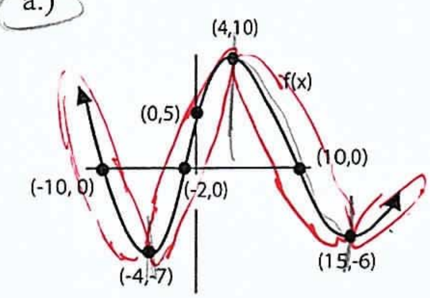
A function is **constant** on an interval I, if for all values of x in I, then the values of $f(x)$ are equal.

Some books will define these intervals to be strictly open. This book does *not*.

Think left to right.

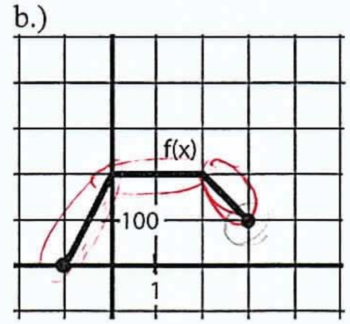
expl 6: For each function below, determine the intervals where the function is increasing, decreasing, or constant. Write your answers in interval notation using square brackets.

a.)



inc: $[-4, 4], [15, \infty)$
dec: $(-\infty, -4], [4, 15]$

b.)



inc: $[-1, 0]$
dec: $[2, 3]$
cnst: $[0, 2]$

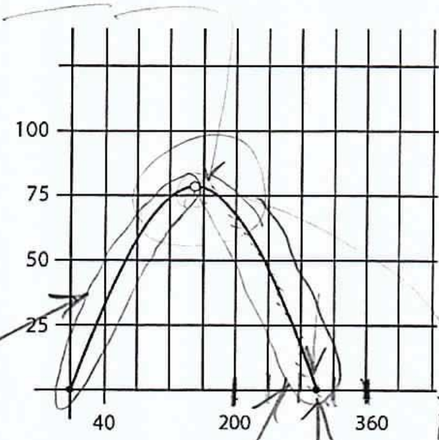
Remember these intervals are *x-values*.

3.3

Holes in graphs:

expl 7: Consider this amended graph from a previous problem. Notice the hole at the top of the parabola.

inc



Here, the (approximated) point at the top (150, 80) is *no* longer considered a maximum. Do you see why? No abs. max

Write the intervals where this graph is increasing or decreasing.

We should *not* include ~~80~~ 150 so use half-open intervals.

dec

$x \approx 300$

inc: $[0, 150)$

dec: $(150, 300]$

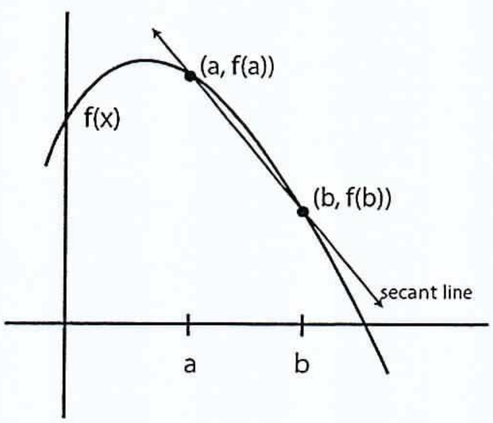
Average Rate of change:

You might recall that the slope of a straight line is the difference of the y-values divided by the difference of the x-values. This ratio tells us how fast y is changing with respect to x, or the **average rate of change**.

Calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

What if we used this to investigate a non-linear function? We can select two points on the graph and find the slope of the line between them (called the **secant line**). We will amend our formula for slope a bit, using function notation.

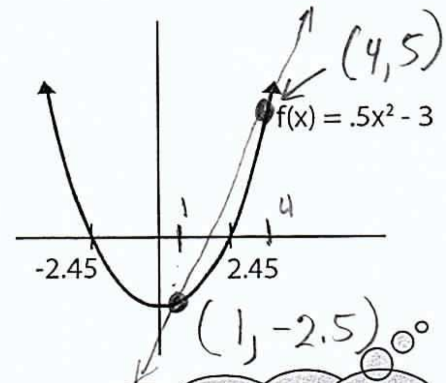


Definition: Average rate of change:

If a and b where $a \neq b$ are in the domain of $f(x)$, then the **average rate of change** of f from a to b is $\dfrac{\Delta y}{\Delta x} = \dfrac{f(b) - f(a)}{b - a}$, $a \neq b$.

expl 8a: For the function pictured to the right, find the average rate of change from 1 to 4.

$a=1$ $\quad b=4$



$$f(x)=0.5x^2-3$$

$$f(1)=0.5(1)^2-3=-2.5$$

$$f(4)=0.5(4)^2-3=5$$

$$\text{avg rate of change} = \frac{f(4)-f(1)}{4-1}$$

$$=\frac{5+{}^{+}2.5}{3}=\frac{7.5}{3}=2.5 \text{ avg rate of change}$$

$$\frac{\Delta y}{\Delta x}=\frac{f(b)-f(a)}{b-a}$$

b.) Do your best to plot the points we are talking about above. Draw the secant line whose slope you have just found. ✓

Do you remember what it means graphically for a slope to be positive or negative?

$16,066,000,000,000

expl 9: The US government's debt is a function of time that is increasing. In 2012, it was $16,066 billion. In 2018, it had grown to $21,516 billion. Find the average rate of change for these years. Describe, in words, what this average tells us about how much the debt grows each year.

debt is in billions of dollars

Can you write $16,066 billion as a plain number?

(time, debt)

(2012, 16,066)

(2018, 21,516)

$$\text{avg rate of change} = \frac{f(b)-f(a)}{b-a}=\frac{21{,}516-16{,}066}{2018-2012}\approx\frac{908}{1}$$

billions of dollars

years

So, avg rate of change is $908 billion.

So, for every year the debt grows by an average of $908 billion.