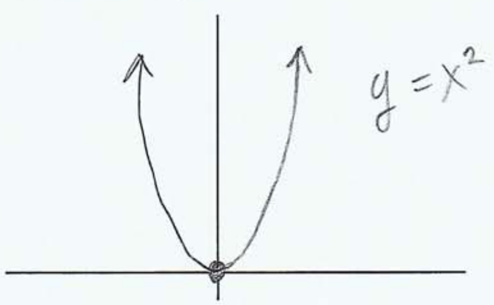
We will investigate the shapes, vertices, orientations, intercepts, and axes of symmetry for these functions.

College algebra
Class notes
Graphs of Quadratic Functions (section 4.3)

Definition: Quadratic Function: A quadratic function is a function that could be written in the form $f(x) = ax^2 + bx + c$ where a is *not* zero. We will also use the (**Vertex**) form $f(x) = a(x-h)^2 + k$ where a is *not* zero. (Completing the square will get us from the first form to the other.)

We will investigate the graphs of quadratic functions such as $y = (x+5)^2$, $f(x) = 4x^2$, and $g(x) = -3(x-4)^2 + 9$. These can be considered transformations on the basic function $y = x^2$. So let's start there…

Graph $y = x^2$ from memory. Recall the nice symmetric shape. Also, notice how its vertex is exactly on the origin.



This shape is called a **parabola**.

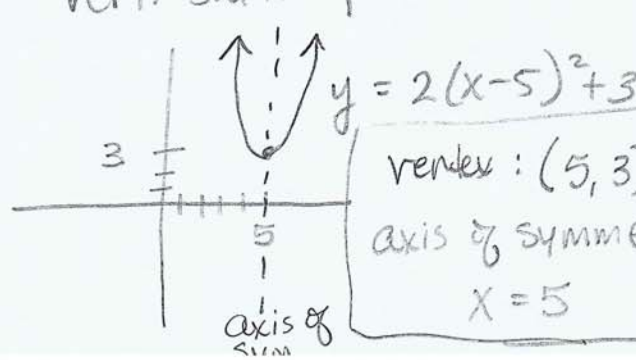
Definition: Vertex and Axis of Symmetry: The **vertex** of a parabola is where it bends. We will use ordered pair notation to write the vertex. The **axis of symmetry** is the vertical line that goes through the vertex. Notice the graph is symmetric about that imaginary line.

When a quadratic function is in the form $f(x) = a(x-h)^2 + k$, you can determine its graph by thinking of the various transformations that you apply to $y = x^2$.

expl 1: Use transformations to describe how the graph of $f(x) = 2(x-5)^2 + 3$ differs from $y = x^2$. From that information, determine the vertex and axis of symmetry of f. Also, draw a quick graph of f.

mother fnc: $y = x^2$
trans #1: $y = (x-5)^2$
hor shift to right 5 units
trans #2: $y = 2(x-5)^2$
vert. stretch by a factor of 2

trans #3: $y = 2(x-5)^2 + 3$
vert. shift up 3 units



vertex: $(5, 3)$
axis of symmetry
$x = 5$

vertex : (h,k)
axis : $x=h$

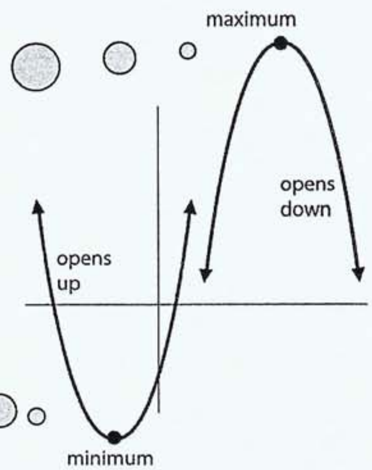
Vertex, Axis of Symmetry, and Orientation of a Parabola:

In general, what would the vertex of the function $f(x) = a(x-h)^2 + k$ be? What is the equation of the axis of symmetry? What about the formula determines if the parabola opens up or down? (This is called **orientation**. Think of a downward orientation as the result of a reflection about the x-axis.)

If $a>0$, parabola opens up.
If $a<0$, it opens down

Remember a, b, and c? it turns out that $h = {-b}/{2a}$ and $k = \left(4ac - b^2\right)/4a$.

A parabola that opens up has a **minimum**. It is called **concave up**. A parabola that opens down has a **maximum**. It is called **concave down**. ✓



Alternative Formula for Vertex of a Parabola: It so happens that if a quadratic function is in the form $f(x) = ax^2 + bx + c$, the x-value of the vertex can be found by calculating $x = \dfrac{-b}{2a}$.

But a vertex has **two coordinates**, an x and a y coordinate. So how would you find the corresponding y-value? Plug it in!

This is usually written as $vertex = \left(\dfrac{-b}{2a}, f\left(\dfrac{-b}{2a}\right)\right)$.

Recall: Finding x- and y-intercepts: To find the x-intercept of a function, substitute 0 for y and solve for x. To find the y-intercept, substitute 0 for x and solve for y. (This is true for any type of function, *not* just quadratic ones.) Finding the x-intercept may require the quadratic formula. That is a good use of the calculator program (or Zero Function under CALC menu) since it gives decimal answers.

For the generic function $y = ax^2 + bx + c$, what will the y-intercept always be? — put $x=0$ in

$y = a(0)^2 + b(0) + c$

$y = c$ (the constant at the end)

Graphing Quadratic Functions:

We will graph quadratic functions by finding and plotting the vertex and y-intercept and using the orientation (and its symmetry) to fill in the rest of the parabola.

We will work with two different forms of a quadratic function, $f(x) = a(x-h)^2 + k$ and $f(x) = ax^2 + bx + c$.

We'll use a function in **Vertex form** on MyMathLab for this example. (#8 in MML homework)

expl 2: Find the vertex and axis of symmetry for a quadratic function (gotten from MML). Determine if the vertex is a minimum or a maximum. Graph the function.

$g(x) = -2(x-2)^2 + 5$

This is in the form $f(x) = a(x-h)^2 + k$. So, pick the vertex out as (h, k). What is h? What is k?

$y = a(x-h)^2 + k$

$h = 2 \quad k = 5$

Answer the following questions.

a.) What is the vertex of the parabola? $(h, k) = (2, 5)$

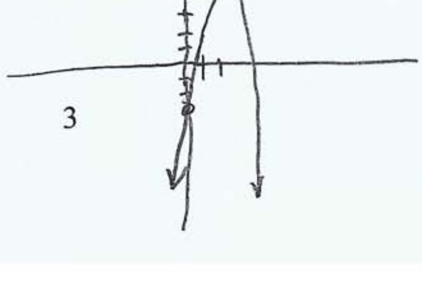
Write the vertex in ordered pair notation. ✓

b.) What is the axis of symmetry? $x = h \rightarrow x = 2$

c.) Is the vertex a maximum or minimum value? In other words, is the parabola concave up or concave down? What is the extremum value?

The value of a (or −2) is negative. So, the parabola opens down – making the vertex a maximum. It is concave down. The maximum (extremum) is $y = 5$ (from the vertex.)

d.) Graph the function.



(y-int is -3 because $g(0) = -3$)

Do you know how to graph a parabola on MyMathLab?

This next example asks the same things. However, the function is given in the $y = ax^2 + bx + c$ form. So, there are two different methods that could be used.

Method 1: The equation is in the form $y = ax^2 + bx + c$. We could convert it to the form $f(x) = a(x-h)^2 + k$ by completing the square. Then we would pick out the vertex and other information.

Method 2: The equation is in the form $y = ax^2 + bx + c$. So, calculate the x-value of the vertex as $x = \frac{-b}{2a}$. How would you find the y-value that goes with it?

Plug that x-value in for x and calculate y.

Recall: Completing the Square:

Completing the square is a technique that forces an expression in the form $ax^2 + bx + c$ into the form $a(x-h)^2 + k$. Particularly, it is the $(x + ?)^2$ part that we want so badly. Before we attack a problem, let's look at why completing the square works.

Look at this pattern: FOIL these problems.

$(x+4)^2 = x^2 + 8x + \underline{\underline{16}}$

$(x+7)^2 = x^2 + 14x + \underline{\underline{49}}$

$(x-5)^2 = x^2 - 10x + \underline{\underline{25}}$

$\frac{8}{2} = 4$

$4^2 = \underline{\underline{16}}$

Recall these are perfect square trinomials.

$\frac{14}{2} = 7, \; 7^2 = \underline{\underline{49}}$

So, we are interested in going from $x^2 + 8x + 16$ back to the $(x+4)^2$ form. But what if we were just given $x^2 + 8x$? How would we figure out the constant that "completed" $x^2 + 8x$ so that we could factor it as $(x+4)^2$? We look at the coefficient of the x-term and guess its relationship to the constant at the end.

In each trinomial above, what is the relationship between the coefficient of the x-term and the constant at the end? The constant is equal to the square of 1/2 of coefficient of x-term.

What would you add to $x^2 + 12x$ so that we could write it as $(x + ?)^2$, and what goes in the parentheses? $\frac{12}{2} = 6 \rightarrow 6^2 = \underline{\underline{36}}$ So $x^2 + 12x + \underline{\underline{36}} = (x + \underline{\underline{6}})^2$

Now that we have the general idea of completing the square, let's use it to rewrite a quadratic function in its alternative form.

expl 3: Rewrite the function below in the **Vertex form** using completing the square. Then answer the questions that follow.

$f(x) = 2x^2 + 8x + 5$

$= 2(x^2 + 4x \; ? \;) + 5$

$= 2(x^2 + 4x + 4) + 5$ ___

$\frac{4}{2} = 2,\; 2^2 = 4$

we just added 2·4 to this so we must subtract it back off at end

Never attempt to complete the square without first factoring out the *x*-squared coefficient from the first two terms.

What you add to complete the square must be countered to keep it equal.

$f(x) = 2(x+2)^2 + 5 - 8$

$f(x) = 2(x+2)^2 - 3$

a.) What is the vertex of the parabola?

$y = a(x-h)^2 + k \rightarrow (h, k) = (-2, -3)$

b.) What is the axis of symmetry?

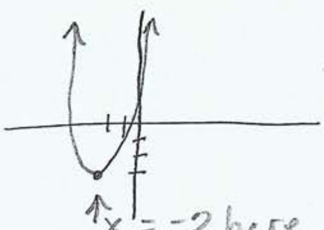
$x = h \rightarrow x = -2$

c.) Is the vertex a maximum or minimum value? In other words, is the parabola concave up or concave down? What is the extremum value?

Our a (or 2) is positive — so parabola opens up making vertex a minimum. The parabola is concave up.

The minimum value is $y = -3$.

d.) Find the intervals over which the function is increasing or decreasing.



$x = -2$ here

dec: $(-\infty, -2]$

inc: $[-2, \infty)$

e.) Just for kicks, let's verify the vertex using the formula $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. This uses the original form given for the function.

$f(x) = 2x^2 + 8x + 5$ $\quad$ $y = ax^2 + bx + c$

$a = 2$
$b = 8$ $\rightarrow x = \frac{-b}{2a} = \frac{-8}{2(2)} = \frac{-8}{4} = -2$

$f(-2) = 2(-2)^2 + 8(-2) + 5 = -3 \rightarrow$ vertex $(-2, -3)$ ✓

Yahoo!

Recall: More about finding x-intercepts:
Do you remember what happens to the solutions of $0 = ax^2 + bx + c$ when $b^2 - 4ac$ (the **discriminant**) is positive, negative, or zero? Are the solution(s) real or imaginary (complex but *not* real)? How many solutions are there? Do you remember? Here is the completed table from a previous section.

If the discriminant is …	**then there will be ________ solution(s).**
positive,	two distinct, real
negative,	no real
zero,	one real, repeated

Real solutions to $0 = ax^2 + bx + c$ are also **x-intercepts** of $y = ax^2 + bx + c$.

expl 5: How many real solutions does the equation $0 = -16x^2 + 100x + 25$ have? (Calculate *only* $b^2 - 4ac$.)

$b^2 - 4ac$
$= 100^2 - 4(-16)(25)$
$= 11600$

So, $b^2 - 4ac > 0$ and so eqn has 2 real zeros (x-intercepts)

Notice that part c below indirectly asks about those x-intercepts.

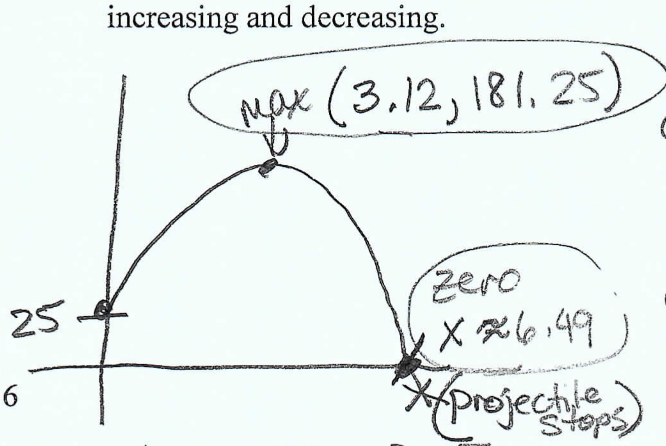
expl 6: A projectile is fired straight upward, with a muzzle velocity of 100 feet per second. The height $h(x)$ of the projectile (in feet) is given by $h(x) = -16x^2 + 100x + 25$ where x is the number of seconds after the projectile is fired. Phrase answers in sentence form with units.

a.) How long does it take the projectile to get to its maximum height?
b.) Find the maximum height of the projectile.
c.) When will the projectile strike the ground?
d.) What is the height when x is 5 seconds?
e.) During which values of x is the projectile going up and during which values of x is the projectile going down? In other words, give the intervals where the function is increasing and decreasing.

domain $[0, 6.49]$

Can you graph the function?

Do it algebraically or graphically.



window: $[0, 7] \times [-50, 200]$

a and b
The projectile reaches its maximum height of 181.25 ft in 3.12 sec.
ⓒ It strikes ground after 6.49 sec.
ⓓ $h(5) \approx 125$ ft
ⓔ inc: $[0, 3.12]$
dec: $[3.12, 6.49]$

expl 7: A cereal company has determined that the revenue $R(x)$ that they make from selling x boxes of cereal is given by $R(x) = 500x - 6x^2$. If they make (and then sell) x boxes, their cost is given by $C(x) = 700 + 8x$. Find a formula for the company's profit if they make and sell x boxes. How many should they make and sell to maximize their profit? What is the maximum profit?

$$P(x) = R(x) - C(x)$$
$$= 500x - 6x^2 - (700 + 8x)$$
$$= 500x - 6x^2 - 700 - 8x$$

Profit is revenue minus cost.

$$P(x) = -6x^2 + 492x - 700$$ (opens down with max)

Vertex: $x = -b/2a = -492/2(-6) = 41$ (x-value of vertex)

$$P(41) = -6(41)^2 + 492(41) - 700 = 9386$$ (y-value of vertex)

To maximize profit, they should make and sell 41 boxes. Their max profit will be \$9386 (assuming units is dollars).

This is an optional application which is very cool but will *not* be represented in the homework.

expl 8: A rock is dropped from a high cliff. The sound of it hitting the ground is heard 2.5 seconds later. How high is the cliff?