Zero Exponent Rule

$a^0 = 1$ (if $a \neq 0$)

We will learn the basics of polynomial functions and their graphs.

College algebra
Class notes
Polynomial Functions (section 5.1)

whole #s: $\{0, 1, 2, 3, \ldots\}$

Definition: Polynomial function:
A polynomial function is a function that can be written in the form
$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$ where $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are real numbers and the exponents are whole numbers.

$a_0 x^0 = a_0(1) = a_0$

No exponents can be fractions or negative numbers.

Do you see how these fit the general formula?

expls: $y = ax^2 + bx + c$

$f(x) = mx + b$

$y = \frac{1}{2}x^5 - 4x^4 + 2x^3 - 7$

$f(x) = \sqrt{5}x^3 + 4x^2 - x + 10$

$g(x) = \sqrt{2}x^2 - 5x^7 + 4x^6 - \sqrt{12}$

$y = (2x+3)(x-1)$

Standard form means the exponents are in decreasing order.

Some counterexamples follow. Can you tell why they are *not* polynomials? How do they *not* fit the definition?

counterexpls: $y = 4x^{-3} + 2x - 7$, $f(x) = \frac{1}{2}\sqrt{x} + 4x^2$, $y = \frac{14x^2 + x}{x^3 - 8}$

$\sqrt{x} = x^{1/2}$

Terminology:

$a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are called **coefficients**

a_0 is called the **constant coefficient** (or **term**)

★ a_n is called the **leading coefficient**

★ $a_n x^n$ is called the **leading term**

★ n is called the **degree** (largest exp.)

x is called the **variable**

Consider $y = 4x^3 - 5x^2 + 8$. Identify these parts.

coeff: 4, -5, 8
$a_0 = 8$
lead coeff: 4
lead term: $4x^3$
degree = 3

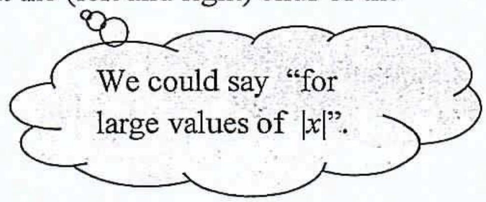
Characteristics of Graphs of Polynomial Functions:

1.) The **domain** will always be "all real numbers". (We have *no* square roots or division by 0 in polynomial functions. So nothing stands in the way of an x value "working" in the function.)

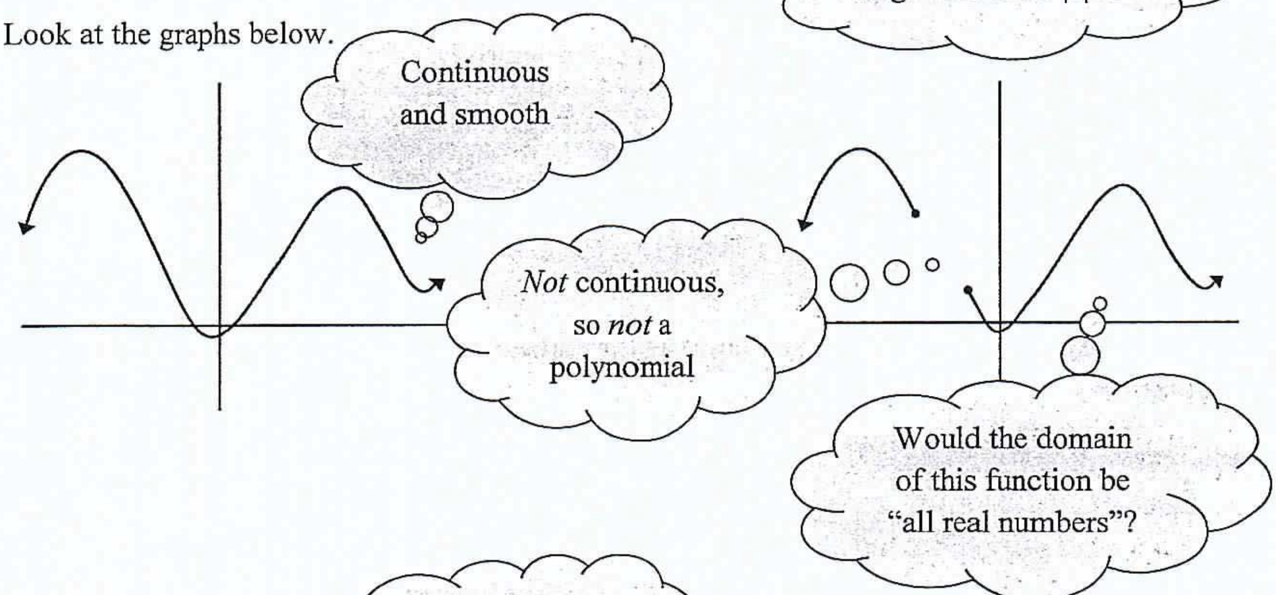
2.) The graph is **continuous**. This means you could trace the whole graph from the left end to the right end without lifting your pencil.

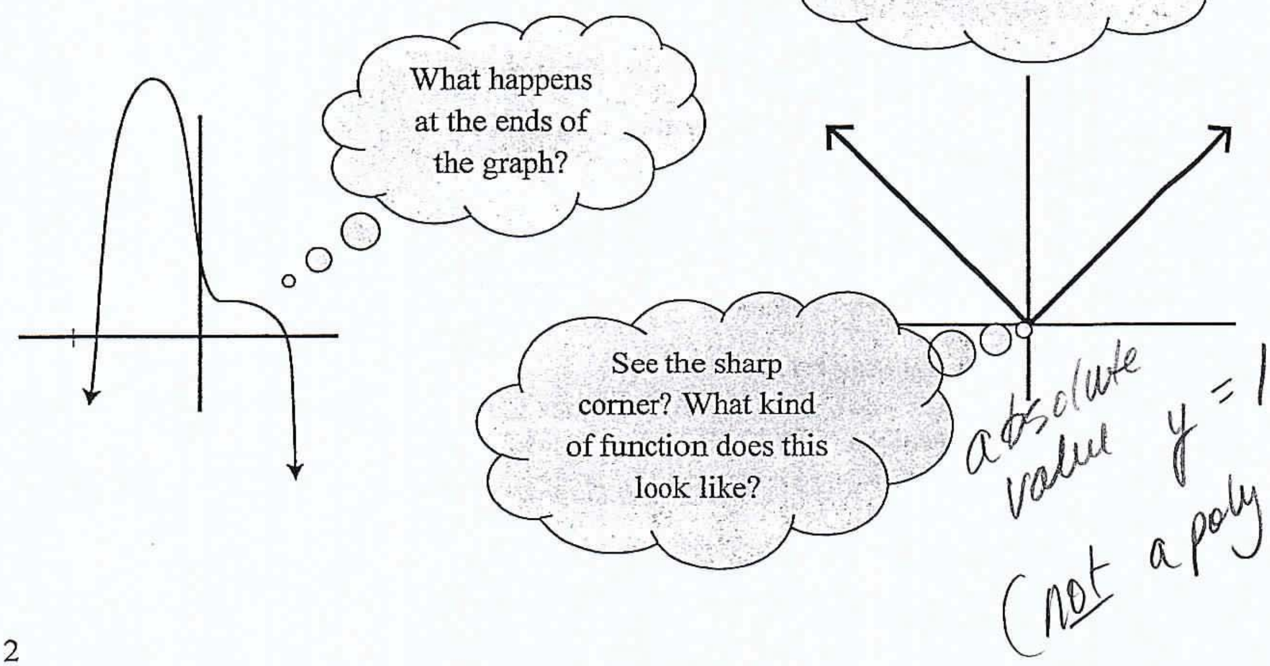
3.) The graph has **no sharp corners**. It is a smooth curve.

4.) The last major characteristic of a polynomial graph is its **end behavior**. End behavior answers the question, "what is happening to the y values at the (left and right) ends of the graph?" We'll investigate end behavior next.



Look at the graphs below.





absolute value $y = |x|$ (not a poly fnc)

Worksheet: Polynomial functions: End behavior:

This worksheet explores the end behavior of the graphs of polynomial functions. We look at what is happening to the y values at the (left and right) ends of the graph. In other words, we are interested in what is happening to the y values as we get really large x values and as we get really small (negative) x values.

Fill in this table as a summary of what you learned from the worksheet. Recall how the leading term alone will determine a polynomial function's end behavior. This is sometimes referred to as the **Leading Term Test**.

	Leading coefficient is negative	**Leading coefficient is positive**
Degree is odd	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to -\infty$ (expl 2)	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to \infty$ (expl 1)
Degree is even	as $x \to -\infty$, $y \to -\infty$ as $x \to \infty$, $y \to -\infty$ (expl 4)	as $x \to -\infty$, $y \to \infty$ as $x \to \infty$, $y \to \infty$ (expl 3)

expl 1: Determine the leading term, leading coefficient, constant term, and the degree of the polynomial.

a.) $r(x) = -5x^4 + 2x^3 - 7$

lead term: $-5x^4$

lead coeff: -5

degree: 4

constant term: -7

Leading term: term with highest exponent
Constant term: no x

b.) $f(x) = 3 - 5x^2 + 9x^3 - 6x^7 + 5x$

lead term: $-6x^7$

lead coeff: -6

degree: 7

const. term: 3

expl 2: Find the end behavior of each function. Write the end behavior using the notation shown in the worksheet "Polynomial functions: End behavior".

a.) $r(x) = -5x^4 + 2x^3 - 7$

as $x \to -\infty$, $y \to -\infty$

as $x \to \infty$, $y \to -\infty$

b.) $f(x) = 3 - 5x^2 + 9x^3 - 6x^7 + 5x$

as $x \to -\infty$, $y \to \infty$

as $x \to \infty$, $y \to -\infty$

Worksheet: Polynomial functions: End behavior 2:
This worksheet will help you practice use the procedure and notation described here.

Definition: Power function: A **power function** is a polynomial function with only one term (called a monomial). In general, we write this as $y = ax^n$ where a is a real number *not* equal to 0 and n is an non-negative integer. Notice that n would be the **degree** of this function.

$\{0, 1, 2, 3, \ldots\}$

Connection Between Polynomial End Behavior and Power Functions:
Since the end behavior of a polynomial function is solely determined by the leading term, we can talk about the end behavior of a polynomial function f by referring to the power function that "resembles the graph of f for large values of $|x|$."

For instance, consider the function from the last page, $r(x) = -5x^4 + 2x^3 - 7$. Which "power function" would the graph of $r(x)$ resemble for large values of $|x|$?

$y = -5x^4$

Again, by "large values of $|x|$", we mean the ends of the graph.

If a function is given in factored form, multiply it all out to see it in standard form.

Zeros of Polynomial Functions:

Recall: Definitions: Zero and x-intercept:
An **x-intercept** is where the graph hits the x-axis. Since the y value is 0 at these points, these are also the x values that make $f(x) = 0$. A real number that makes $f(x) = 0$ is called a **real root** or **real zero**.

expl 3) Use substitution to determine if the values -5 and 3 are zeros of the function $g(x) = 3x^2 + 9x - 30$.

Find $g(-5)$. See if I get 0 out.

Use the TABLE function on the calculator.

$\left.\begin{array}{l} g(-5) = 0 \\ g(3) = 24 \end{array}\right\} \rightarrow$ So, -5 is a zero of $g(x)$ but 3 is not.

expl 4: Consider the function $f(x) = x(x-3)^2(x+1)$. Algebraically solve $f(x)=0$ to find the zeros of this function.

$0 = x \cdot (x-3)^2 (x+1)$

$x = 0$ or $(x-3)^2 = 0$ or $x+1=0$

$\sqrt{(x-3)^2} = \sqrt{0}$

$x - 3 = 0$

$x = 3$

$x = -1$

$\boxed{\text{Zeros: } x = -1, 0, 3}$

What did you get? Notice how these zeros are tied directly to the factors of $f(x)$. The following theorem spells this relationship out.

Zero / Factor Theorem:
Let f ~~be a polynomial~~ function and r be a real number in its domain. The expression $x - r$ is a factor of f if and only if r is a zero of f.

What does "if and only if" mean?

Definition: Multiplicity of a zero: The multiplicity of a zero (or root) is the number of times its corresponding factor appears in the factored form of the polynomial.

The book uses this more exacting definition.

Definition: Real zero of multiplicity m: If $(x-r)^m$ is a factor of a polynomial f and $(x-r)^{m+1}$ is *not* a factor of f, then r is called a real zero of multiplicity m.

expl 5: In the last example, we found the zeros of this function. State each zero's multiplicity.

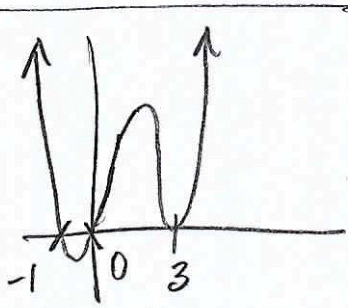
$f(x) = x(x-3)^2(x+1)$

when $r = -1$,
$x - r = x - -1$
$= x + 1$

Zero	Multiplicity
−1	1
0	1
3	2

How many times does each zero's corresponding factor appear?

It turns out that the multiplicity of a zero impacts how the graph appears at that x-intercept. Graph this function on the standard window.



The graph acts differently at the zeroes of −1 and 0 than it does at the zero of 3. Do you see it?

We have this result.

Graphs of Zeroes and Their Multiplicities:

If r is a zero of odd multiplicity, the graph will cross through the x-axis at r.

If r is a zero of even multiplicity, the graph will only touch the x-axis at r.

Odd and even multiplicity simply refers to the number itself. Is it odd or even?

expl 6: Find a polynomial function of degree 3 whose real zeros are -3, 0, and 4. Simplify your answer; leave it in standard form.

How do we use the Zero / Factor Theorem?

$$f(x) = (x+3)\cdot x \cdot (x-4)$$

$$= (x^2+3x)(x-4)$$

$$= x^3 + 3x^2 - 4x^2 - 12x$$

$$\boxed{f(x) = x^3 - x^2 - 12x}$$

expl 7: Find a polynomial function of degree 3 whose real zeros are -3, 0, and 4 and also passes through the point (5, 160). Simplify your answer; leave it in standard form.

Once we get a basic $f(x)$, we want y to be 160 when x is 5. How do we force that?

$$f(x) = x^3 - x^2 - 12x$$

$$f(5) = 5^3 - 5^2 - 12(5)$$

Now, $f(5) = 40$

But I want it to equal 160 ∴

So $\boxed{f(x) = 4(x^3 - x^2 - 12x)}$

Check: $f(5) = 4(5^3 - 5^2 - 12 \cdot 5)$

$= 160$ ✓

Turning Points:

Definition: Turning point: A **turning point** in the graph is where it changes from increasing to decreasing or vice versa.

Turning points are explored in calculus.

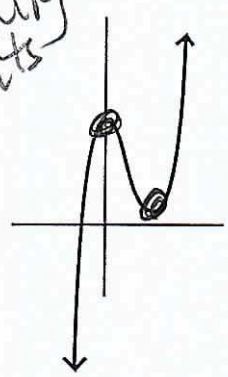
It happens to be true that a degree n polynomial can have no more than $n-1$ turning points. (Relatedly, the graph can have no more than n zeros.)

This information is really helpful when you graph by hand.

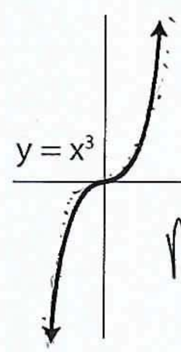
expl 8: Mark the turning points, if they exist, on the graphs below.

a.)

2 turning points



b.)



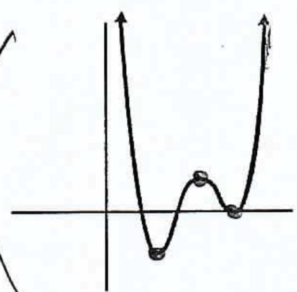
no turning points

A turning point is where a graph changes from increasing to decreasing or vice versa.

Verifying a Complete Graph:
Knowing what we know (the end behavior, the maximum number of zeros, and the maximum number of turning points that a function can have) helps us graph it by hand or determine if we have a complete graph on the calculator.

For instance, let's say we graph $y = x^4 - 11x^3 + 42x^2 - 64x + 32$. The graph is shown below. Using what you learned from this section, explain how you know it has to be a complete graph.

$y = x^4 - 11x^3 + 42x^2 - 64x + 32$



A complete graph shows all intercepts, turning points, and the end behavior.

degree = 4 = n

It can have no more than

3 turning points.

degree minus 1

n-1

done in class (not collected)

Polynomial functions: End behavior NAME:

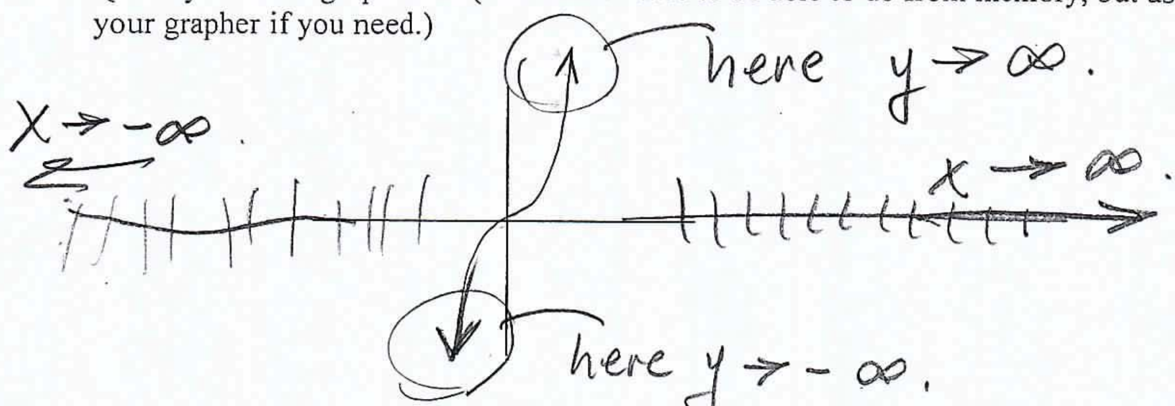
We are looking at the end behavior of polynomial graphs, i.e. what is happening to the y values at the (left and right) ends of the graph.

★ In other words, we are interested in what is happening to the y values as we get really large x values and as we get really small (negative) x values.

right end left end

Recall a polynomial function is one that can be written in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where n is the degree of the polynomial (must be a nonnegative integer), $a_n, ..., a_0$ are the coefficients, and a_n is called the leading coefficient.

To get an understanding of how we will denote end behavior, let's look at $y = x^3$. Quickly sketch a graph of it. (It would be nice to be able to do from memory, but use your grapher if you need.)

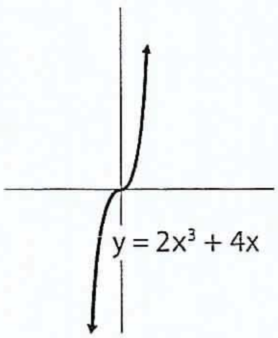


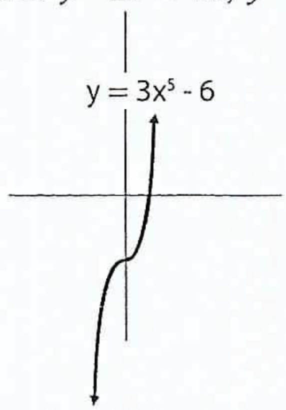
Notice how the y values soar off toward infinity on the right end (as x values get really big) and the y values soar off toward negative infinity on the left end (as x values get really small).

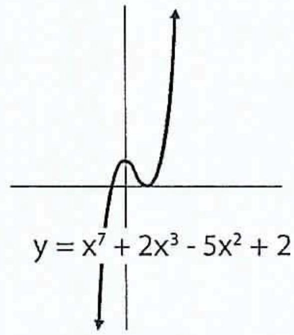
We denote this by writing "as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$". This is read "as x approaches negative infinity, y approaches negative infinity and as x approaches (positive) infinity, y approaches (positive) infinity". Notice the first part of this talks about the left end of the graph and the second part of this talks about the right end of the graph.

This worksheet will guide you through looking at the end behaviors of several polynomial functions. At the end, we will generalize about all polynomial functions. Graphs are provided but if you want to graph along, a good window for all the graphs will be [-10, 10] x [-25, 25] unless stated otherwise.

1. Below are the graphs of $y = 2x^3 + 4x$, $y = 3x^5 - 6$, and $y = x^7 + 2x^3 - 5x^2 + 2$.





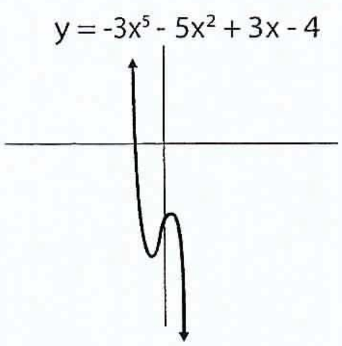


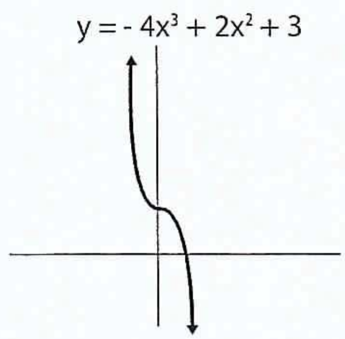
What is the end behavior of all three graphs above? Use the notation demonstrated on the first page. Remember the first part describes the left end of the graph and the second part describes the right end of the graph.

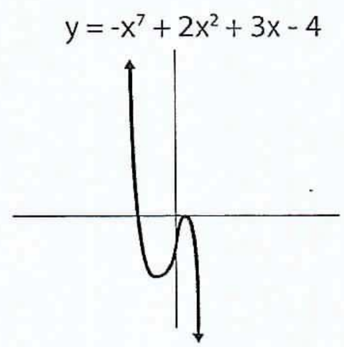
left : as $x \to -\infty$, $y \to -\infty$

right: as $x \to \infty$, $y \to \infty$

2. Below are the graphs of $y = -3x^5 - 5x^2 + 3x - 4$, $y = -4x^3 + 2x^2 + 3$, and $y = -x^7 + 2x^2 + 3x - 4$.







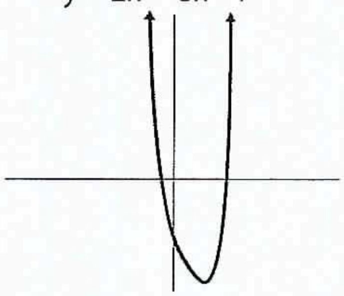
What is the end behavior of all three graphs above? Use the notation demonstrated on the first page. Remember the first part describes the left end of the graph and the second part describes the right end of the graph.

left: as $x \to -\infty$, $y \to \infty$

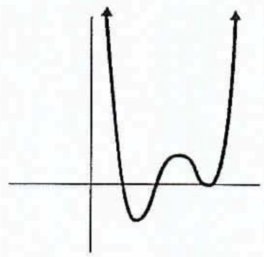
right: as $x \to \infty$, $y \to -\infty$

3. Below are the graphs of $y = 2x^4 - 6x - 4$, $y = x^4 - 11x^3 + 42x^2 - 64x + 32$, and $y = x^6 + 5x^5 - x^4 - 21x^3$. If you are graphing on your calculator, the window for the third one should be set to [-10, 10] x [-50, 75].

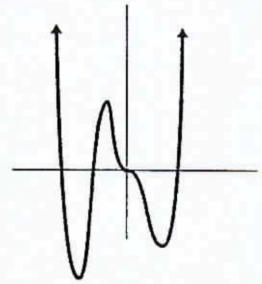
$y = 2x^4 - 6x - 4$



$y = x^4 - 11x^3 + 42x^2 - 64x + 32$



$y = x^6 + 5x^5 - x^4 - 21x^3$



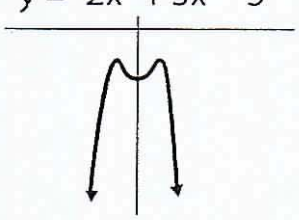
What is the end behavior of all three graphs above? Use the notation demonstrated on the first page.

left: as $x \to -\infty$, $y \to \infty$

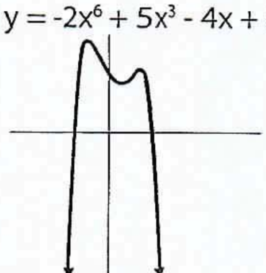
right: as $x \to \infty$, $y \to \infty$

4. Below are the graphs of $y = -2x^4 + 3x^2 - 5$, $y = -2x^6 + 5x^3 - 4x + 8$, and $y = -x^4 + 3x^3 - 4x^2 + 7$.

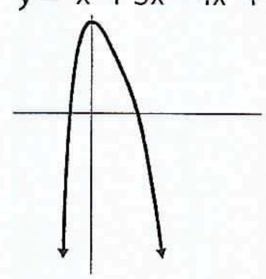
$y = -2x^4 + 3x^2 - 5$



$y = -2x^6 + 5x^3 - 4x + 8$



$y = -x^4 + 3x^3 - 4x^2 + 7$



What is the end behavior of all three graphs above? Use the notation demonstrated on the first page.

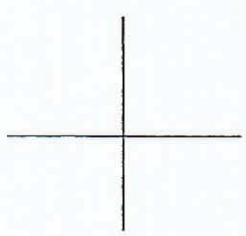
left: as $x \to -\infty$, $y \to -\infty$

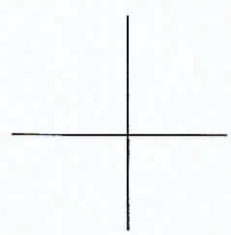
right: as $x \to \infty$, $y \to -\infty$

5. Questions one through four gave you three examples of each kind of (polynomial) end behavior. **The two things that determine the end behavior of a polynomial are the degree (whether it's even or odd) and the leading coefficient (whether it's positive or negative). Look over your work for questions one through four to verify this.** Use the table below to summarize the end behaviors of polynomials. Use the formal notation that is used throughout the worksheet.

	Leading coefficient is negative	**Leading coefficient is positive**
Degree is odd		
Degree is even		

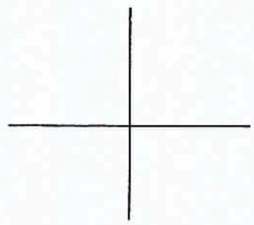
6. **Way to remember end behavior of polynomials:** I remember end behaviors by keeping pictures of $y = x^2$ and $y = x^3$ in my head. Draw these two functions now. (You should be able to do so from memory.) What are the end behaviors of these functions? Use the formal notation that is used throughout the worksheet.



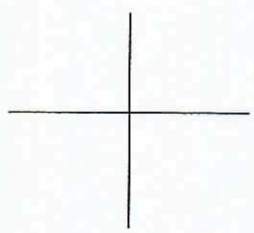


I remember that all polynomials of even degree and positive leading coefficient have the same end behavior as $y = x^2$. And all polynomials of odd degree and positive leading coefficient have the same end behavior as $y = x^3$.

For polynomials of even degree and negative leading coefficient, I picture $y = -x^2$ which is a reflection of $y = x^2$ about the x-axis. Draw $y = -x^2$ now, using this information. What is the end behavior of $y = -x^2$? **All polynomials of even degree and negative leading coefficient share this same end behavior.**



For polynomials of odd degree and negative leading coefficient, I picture $y = -x^3$, which is a reflection of $y = x^3$ about the x-axis. Draw $y = -x^3$ now, using this information. What is the end behavior of $y = -x^3$? **All polynomials of odd degree and negative leading coefficient share this same end behavior.**



Use the notation described here for end behavior of polynomial functions. You will also be introduced to shorthand notation in the notes and textbook.