We will look at the graphs of these functions, exploring their domain and end behavior.

College algebra
Class notes
Rational Functions with Vertical, Horizontal, and Oblique Asymptotes (section 5.5)

Definition: Rational Function: A rational function is a function that can be written in the form $r(x)=\dfrac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial functions and $q(x)\neq 0$.

The **domain of any rational function** is all real numbers except the numbers that make the denominator zero or where $q(x)=0$.

dom: $(-\infty,-6)\cup(-6,\infty)$

examples: $y=\dfrac{3x^2+5x-4}{x+6}$

What are these functions' domains?

$$y=\frac{5t^3-9t^2}{6t-5+3t^2}$$

$$f(x)=\frac{5x+4}{7x^{10}-35x^7+7}$$

How do these **not** fit the definition?

counterexamples: $y=\dfrac{\sqrt{3x^2-4x+7}}{5x^2-3x}$, $\qquad h(t)=\dfrac{5t^{-4}+7}{6t-5t^6+81}$

We will study vertical and horizontal/oblique asymptotes of these functions. Knowing where the asymptotes lie and how to find x and y-intercepts will help us understand their graphs.

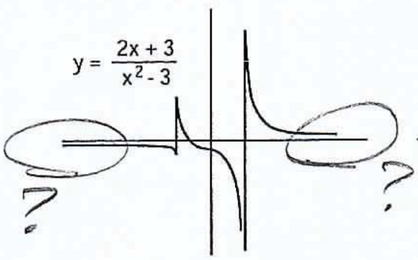
Worksheet: Rational functions: Vertical Asymptotes:
This worksheet discusses finding vertical asymptotes by determining where the denominator is zero. It also looks at the exception to the rule, where holes in the graph will occur.

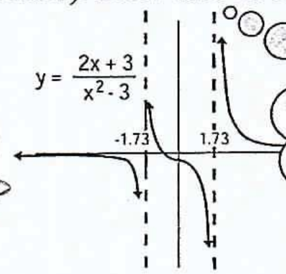
Vertical Asymptotes: Vertical asymptotes and domain are related. The domain is essentially all real numbers except those that make the bottom zero. Often, these are also where the vertical asymptotes lie. If the numerator and denominator have a common factor, then the resulting zero is *not* a vertical asymptote, but rather a hole in the graph. This is discussed on the preceding worksheet.

Put parentheses around the entire top and the entire bottom when graphing.

Drawing Graphs: Consider the rational function $y = \dfrac{2x+3}{x^2-3}$. Graph it in the standard window.

Older calculators' graphs will probably look like the left picture below. In "connect" mode, the calculator plots vertical lines where there don't belong. When you copy it, draw these asymptotes as dashed lines and label them. Newer calculators may graph it correctly as the right picture below (but without dashed lines). Draw arrows on all the ends.





Draw arrows and dashed lines.

How are domain and vertical asymptotes related?

expl 1: Find the domain and determine the vertical asymptotes (or holes in the graph) for the functions below.

a.) $f(x) = \dfrac{x^2-4}{(x+5)(x+2)}$

$-5 \quad -2 \quad 0$

Can you factor a difference of squares? $a^2 - b^2 = ??$

domain: solve "bottom = 0"

$(x+5)(x+2) = 0$

$x+5=0$ or $x+2=0$

$x=-5 \qquad x=-2$

$a^2-b^2 = (a+b)(a-b)$

domain: all real numbers except -5 and -2.
or $(-\infty,-5) \cup (-5,-2) \cup (-2,\infty)$

VA or hole? $f(x) = \dfrac{x^2-2^2}{(x+5)(x+2)} = \dfrac{\cancel{(x+2)}(x-2)}{(x+5)\cancel{(x+2)}} = \dfrac{x-2}{x+5}$

$\boxed{\text{VA: } x=-5 \qquad \text{hole: } x=-2}$

b.) $y = \dfrac{4}{2x^3-x^2-3x}$

domain: "bottom = 0"

$2x^3-x^2-3x=0$

$x(2x^2-x-3)=0$

$x(2x-3)(x+1)=0$

$x=0$ or $2x-3=0$ or $x+1=0$

$x=3/2 \qquad x=-1$

dom: all real #s except $-1, 0,$ and $\frac{3}{2}$.
or $(-\infty,-1) \cup (-1,0) \cup (0,\frac{3}{2}) \cup (\frac{3}{2},\infty)$

VA or holes? $y = \dfrac{4}{x(2x-3)(x+1)}$

(no common factors $\rightarrow$ no holes)

$\boxed{\text{VA: } x=-1, 0, 3/2}$

Horizontal and Oblique Asymptotes (and Backbones):
Horizontal and oblique asymptotes are essentially the end behavior of rational functions. They tell us what is happening at the left and right ends of the graph.

We will see how the graph approaches a horizontal line or a slanted (oblique) line at the left and right ends. On occasion, we will see a graph approach a curved (for instance, parabolic) asymptote such as $y = x^2 + 4$.

deg = 1

Consider the previous graph of $y = \frac{2x+3}{x^2-3}$. What does it look like the y values are approaching on the left and right ends of the graph?

$y \to 0$ at ends of graph.

deg = 2

Worksheet: Rational functions: Horizontal and Oblique Asymptotes:
This worksheet will show you the procedures and many examples for finding horizontal and oblique asymptotes. This worksheet does *not* differentiate between oblique asymptotes and backbones, as discussed below.

Horizontal and Oblique Asymptotes (and Backbones):
The horizontal or oblique asymptotes depend on the degrees of the polynomials on top and bottom that make up the rational function.

We will see three cases: 1. degree on top is **equal to** degree on bottom,
2. degree on top is **less than** degree on bottom, and
3. degree on top is **greater than** degree on bottom.

1. If the degree on top is **equal to** the degree on bottom, then divide the leading coefficients to determine the horizontal asymptote. The horizontal asymptote will be "y = this quotient".

2. If the degree on top is **less than** the degree on bottom, then $y = 0$ is the horizontal asymptote.

3. If the degree on top is **greater than** the degree on bottom, then there is *no* horizontal asymptote. If the degree on top is only one more than the degree on bottom, there is an oblique (or slanted line) asymptote. If the degree on top is two or more than the degree on bottom, you can find the asymptote (called a backbone) though it is *not* technically oblique (as it is *not* a line). We find these asymptotes by dividing the top by the bottom using polynomial long division.

[Polynomial long division will be covered later in this section.]

A note about graphs and asymptotes: Vertical asymptotes occur where the function is undefined. So the graph will **never** cross a vertical asymptote. Horizontal or oblique asymptotes (or backbones) are simply what the graph approaches at the ends of the graph. A graph can cross through a horizontal or oblique asymptote but it does not have to.

All asymptotes are *not* part of the function so should be drawn as dashed lines.

expl 2: Find the horizontal or oblique asymptote for the functions below. Be sure to write it in "$y = \ldots$" form. How do you know if the asymptote is horizontal or oblique?

a.) $y = \dfrac{1 \cdot x^3}{2x^3 - x^2 - 3x}$ ← deg = 3 ← deg = 3

HA: $y = \frac{1}{2}$

Graph the function on the window [-1000, 1000] x [-1,1] to check. Trace to the ends of the graph if you need.

b.) $g(x) = \dfrac{x^2 + 4x}{x^3 + 5x}$ ← deg = 2 ← deg = 3

HA: $y = 0$

expl 3: Since the degree on top is greater than the degree on bottom by one, we know the following function has an oblique asymptote. Find it.

$g(x) = \dfrac{x^2 + 4x - 1}{x + 3}$ ← deg on bottom = 1

deg on top = 2

Perform polynomial long division. It is discussed on the next page.

The oblique asymptote will be $y = \textit{quotient}$.

Polynomial Long Division: We can divide a polynomial *P(x)* by another polynomial *d(x)* to obtain a quotient *Q(x)* and remainder *R(x)*. We could say that $P(x) = d(x) \cdot Q(x) + R(x)$.

This is exactly like when we divide plain old numbers. Divide 73 by 5, and we get a quotient of 14 with a remainder of 3.

We could write $73 = 5 \cdot 14 + 3$.

Recall how we divide plain old numbers by long division. In fact, do the long division below to recall the steps.

$$\begin{array}{rl} 14 & \leftarrow \text{quotient} \\ 5\overline{)73} & \\ -5\downarrow & \\ \hline 23 & \\ -20 & \\ \hline 3 & \leftarrow \text{remainder} \end{array}$$

Polynomial long division mimics the steps here.

dividend: top
divisor: bottom

Now we will do the long division to find the quotient of $\dfrac{x^2+4x-1}{x+3}$. I show the steps of the division problem above so that you can see how polynomial long division is analogous to it.

Polynomial Long Division:

Step 1: Write it with a division symbol. Look only at the leading terms of the divisor and the dividend (double underlined below). Ask yourself, what times the "x" would make "x^2"? This is similar to asking "how many times does 5 go into 7" in the division problem below on the right.

$\underline{\underline{x}}+3\overline{)\underline{\underline{x^2}}+4x-1}$	$5\overline{)73}$

Step 2: You answer "x". So you write that on top of the division symbol, preferably above the "x^2" term.

$\begin{array}{r} x \\ \underline{\underline{x}}+3\overline{)\underline{\underline{x^2}}+4x-1} \end{array}$	$\begin{array}{r} 1 \\ 5\overline{)73} \end{array}$

Step 3: You multiply x by the divisor "$x + 3$" and write that below the dividend. You then subtract that from the dividend's "$x^2 + 4x$".

$\begin{array}{rl} & x \\ \underline{\underline{x}}+3) & \overline{\underline{\underline{x^2}}+4x-1} \\ & \underline{-\left(x^2+3x\right)} \\ & 1x \end{array}$	$\begin{array}{rl} & 1 \\ 5) & \overline{73} \\ & \underline{-5} \\ & 2 \end{array}$ Notice that is exactly what we did here.

Step 4: You drop the next term down from the dividend, the "–1". Looking at the double underlined terms, ask yourself what times the "x" makes "$1x$"?

$\begin{array}{rl} & x \\ \underline{\underline{x}}+3) & \overline{x^2+4x-1} \\ & \underline{-\left(x^2+3x\right)} \\ & \underline{\underline{1x}}-1 \end{array}$	$\begin{array}{rl} & 1 \\ 5) & \overline{73} \\ & \underline{-5} \\ & 23 \end{array}$ We drop this 3 and repeat.

Step 5: You answer "positive 1" so you write "+ 1" in the quotient space.

$x+1$ $x+3\overline{)\;x^2+4x-1}$ $\underline{-(x^2+3x)}$ $1x-1$	14 $5\overline{)\;73}$ $\underline{-5}$ 23

Step 6: You multiply just the 1 by the divisor "$x + 3$" and write that below the "$1x - 1$". You then subtract like in step 3.

$x+1$ $x+3\overline{)\;x^2+4x-1}$ $\underline{-(x^2+3x)}$ $1x-1$ $\underline{-(1x+3)}$ -4	14 $5\overline{)\;73}$ $\underline{-5}$ 23 $\underline{-20}$ 3 You are done because 3 is less than 5.

Step 7: You are done when the degree of the remainder (which is the bottom line) is less than the degree of your divisor. The analog to this in the problem on the right is that you stop when that number is less than your divisor.

Return to expl 3: For our purposes in this section, the oblique asymptote we are after is the quotient. Write the oblique asymptote of $y = \dfrac{x^2+4x-1}{x+3}$. Then graph ~~both~~ the original function and its asymptote. The standard window should be fine.

OA: $y = x+1$

Synthetic division can be performed in place of long division when the divisor is in the form $x - c$. I am *not* including that procedure here.

expl 4: Find the oblique asymptotes (or backbone for part *c*) for the functions below.

a.) $y = \dfrac{2x^3 - 4x}{x^2 - x}$ $\leftarrow$ deg = 3, $\leftarrow$ deg = 2

divisor, deg = 2; dividend

$$\begin{array}{r} 2x + 2 \\ x^2 - x \overline{)\, 2x^3 + 0x^2 - 4x} \\ -(2x^3 - 2x^2) \quad \\ \hline 2x^2 - 4x \\ -(2x^2 - 2x) \\ \hline -2x \end{array}$$

$-2x \leftarrow$ deg = 1

$\boxed{\text{OA}: y = 2x + 2}$

b.) $f(x) = \dfrac{5x^3 - x^2 + x - 1}{x^2 - x + 2}$

deg = 2

$$\begin{array}{r} 5x + 4 \\ x^2 - x + 2 \overline{)\, 5x^3 - x^2 + x - 1} \\ -(5x^3 - 5x^2 + 10x) \quad \\ \hline 4x^2 - 9x - 1 \\ -(4x^2 - 4x + 8) \\ \hline -5x - 9 \end{array}$$

$-5x - 9 \leftarrow$ deg = 1

$\boxed{\text{OA}: y = 5x + 4}$

c.) $y = \dfrac{x^3 + 1}{x}$

deg = 1

$$\begin{array}{r} x^2 \\ x \overline{)\, x^3 + 1} \\ -x^3 \quad \\ \hline 1 \end{array}$$

$1 \leftarrow$ deg = 0

$\boxed{\text{backbone}: y = x^2}$

Graph Rational Functions:

1. Find the real zeros of the denominator. Use this to determine the domain and possible vertical asymptotes. Sketch vertical asymptotes with dashed lines.
2. Factor the top and bottom of the function. If there are common factors on top and bottom, identify holes in the graph. Holes replace vertical asymptotes at these values.
3. Find the horizontal or oblique asymptote (or backbone). Sketch it with a dashed line.
4. Find the x- and y-intercepts, if they exist.

expl 5: Consider the function here. Find the following and then identify the correct graph.

$$y = \frac{2x^3 - 4x}{x^2 - x}$$

a.) Find the domain.

domain: bottom = 0

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or $x - 1 = 0$

$x = 1$

$\boxed{\text{domain: all real numbers except 0 and 1}}$

b.) Find the vertical asymptotes and/or holes in the graph.

$$y = \frac{\cancel{2x}(x^2-2)}{\cancel{x}(x-1)}$$ → common factor of x means a $\boxed{\text{hole at } x=0}$

$$y = \frac{2(x^2-2)}{x-1}$$ → $\boxed{VA: x = 1}$

$x - 1 = 0$

$x = 1$

c.) Find the horizontal or oblique asymptote. (We did this in example 4.)

OA: $y = 2x + 2$

d.) Find the x- and y-intercepts, if they exist.

x-int: $0 = \dfrac{2x^3 - 4x}{x^2 - x}$

$0 = 2x^3 - 4x$

$0 = 2x(x^2 - 2)$

$\cancel{2x = 0}$ or $x^2 - 2 = 0$

$\cancel{x = 0}$ $\quad x^2 = 2 \quad x = \pm\sqrt{2}$

y-int: $y = \dfrac{2x^3 - 4x}{x^2 - x}$

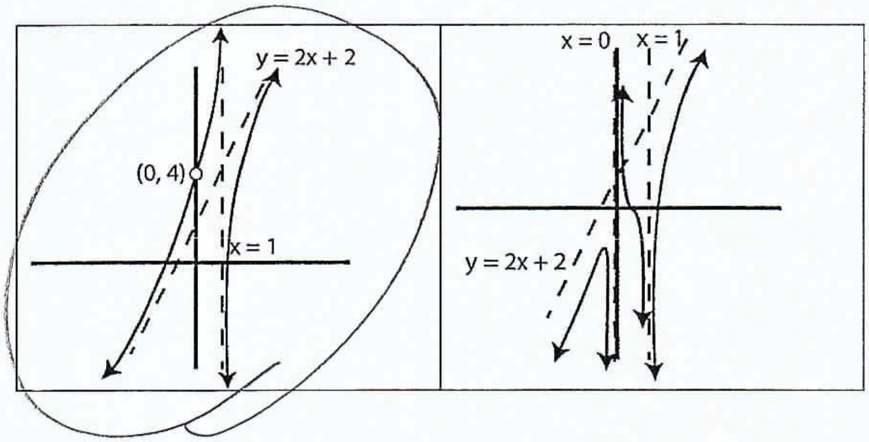
$y = \dfrac{2 \cdot 0^3 - 4 \cdot 0}{0^2 - 0}$

$y = \dfrac{0}{0}$ undefined

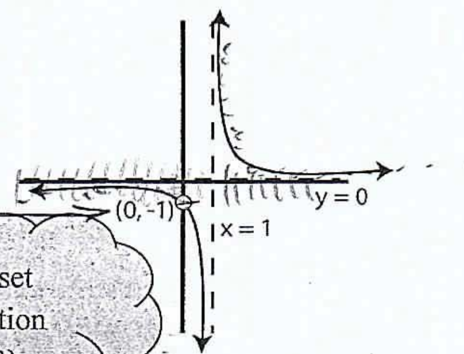
$\boxed{\text{No y-int}}$

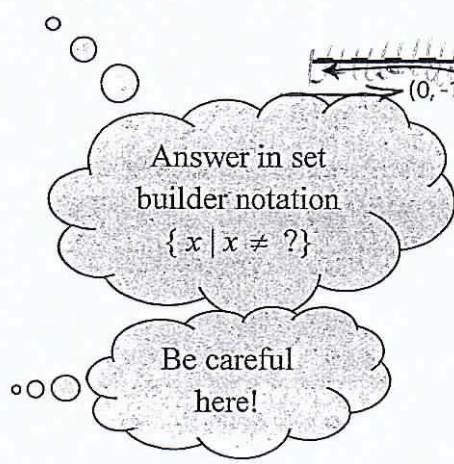
expl 5 continued:

e.) Which of the following is the graph of this function?



expl 6: To the right is the graph of a rational function. Answer the following questions.





a.) What is the domain of the function?

$\{x \mid x \neq 0, 1\}$

b.) What is the range of the function?

$\{y \mid y \neq -1, 0\}$

c.) What are the (x and y) intercepts?

none

d.) What is the horizontal asymptote, if it exists?

$y = 0$

e.) What is the vertical asymptote?

$x = 1$

f.) What is the oblique asymptote, if it exists?

none.

expl 7: Consider the rational function below.

$$g(x) = \frac{3x^2 + 2x - 8}{2x^2 + x - 6}$$

Factor both top and bottom.

a.) Find the vertical asymptote(s) of the function. Also, indicate where any holes in the graph would be.

$$g(x) = \frac{(3x-4)\cancel{(x+2)}}{(2x-3)\cancel{(x+2)}}$$

$\boxed{\text{hole } x = -2}$

$\boxed{\text{VA: } x = 3/2}$

$\downarrow$

Solve $2x - 3 = 0$

$x = 3/2$

b.) What is the horizontal asymptote, if it exists?

$$\boxed{y = \frac{3}{2}}$$

c.) What is the oblique asymptote, if it exists?

none.