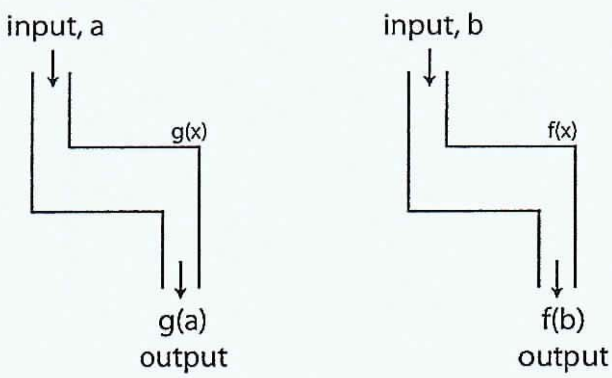
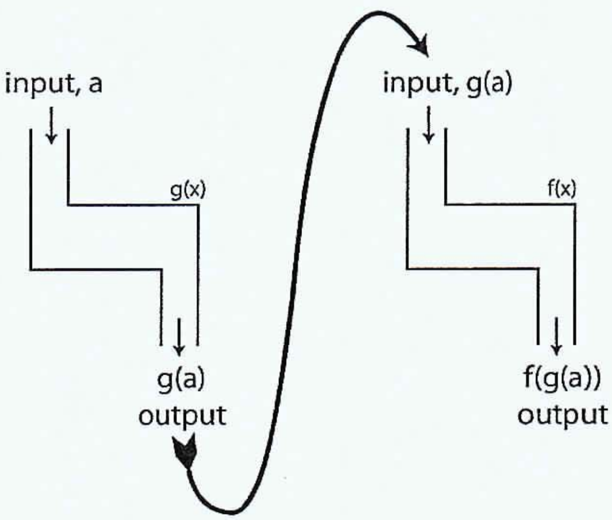
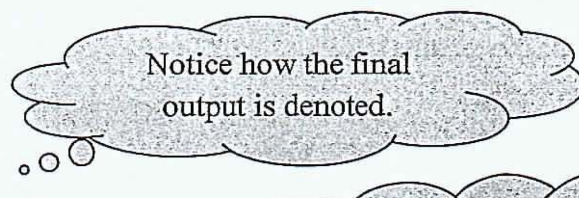
College algebra
Composition of Functions
Section 6.1

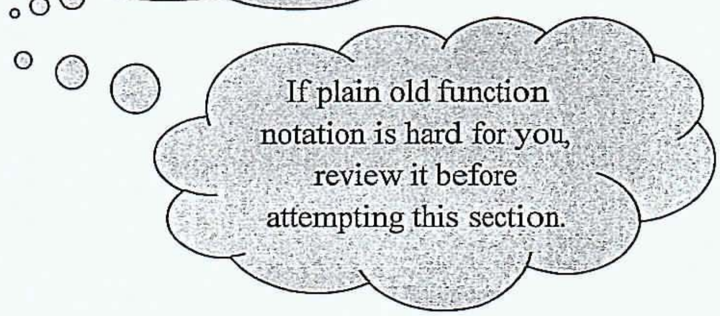
Composition of functions: It will help if you think of functions as machines that take an input and produce an output. Consider two functions, $f(x)$ and $g(x)$, pictured below.



Let's suppose we apply the function g to a number, **and then** put that output into the function f. That is the idea of composition and is pictured below.







We will start our examples with a situation that helps justify why you would ever want to do this composition stuff.

expl 1: A popular umbrella manufacturer has this information.

We can calculate $P(x) = 2x^2 - 3x$, where $P(x)$ represents the monthly profit of the company (in dollars) and x represents the number of umbrellas sold that month, $x \geq 0$.

We can calculate $U(t) = 3t + 4$, where $U(t)$ represents the number of umbrellas sold per month and t represents the number of days it rains that month, $0 \leq t \leq 31$.

We are interested in how profit is related to the number of days of rain. Let's find the equation. But first, we'll investigate the situation with a specific example.

Specific example: Say we have had 12 days of rain this month. Find the number of umbrellas we can expect to sell and then the profit this should yield.

1. Use $U(t)$ to find the number of umbrellas sold...

2. Now find the profit we make...

$U(t) = 3t + 4$

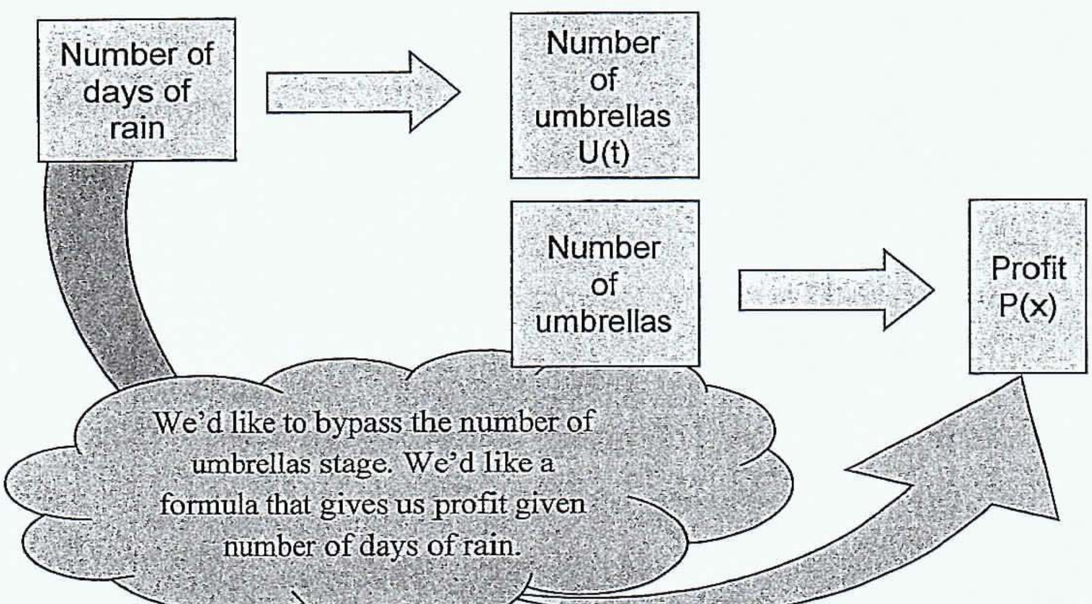
$U(12) = 3 \cdot 12 + 4 = 40$ umbrellas

$P(x) = 2x^2 - 3x$

$P(40) = 2 \cdot 40^2 - 3(40) = \$3,080$

If it rains 12 days in a month, we should make a profit of \$3,080.

Notice how this was a two-step process. Composition of functions can shortcut these steps. I have illustrated this below.



$(3t+4)^2 = (3t+4)(3t+4)$
$= 9t^2 + 12t + 12t + 16$
$= 9t^2 + 24t + 16$

Let's find the relationship between days of rain and profit.

We know $P(x) = 2x^2 - 3x$ where x represents the number of umbrellas.

But $U(t)$ also represents the number of umbrellas. So we can substitute $U(t)$ in for x to find
$P(x) =$ $P(U(t))$.

Apply the rule of $P(x)$ to the number $3t + 4$.

Do it now to make a new function that tells us the profit given the number of days of rain.

Recall, t = # of days of rain.
Find $P(U(t))$

$$= P(3t+4)$$
$$= 2(3t+4)^2 - 3(3t+4)$$
$$= 2(9t^2 + 24t + 16) - 3(3t+4)$$
$$= 18t^2 + 48t + 32 - 9t - 12$$

$$\boxed{P(U(t)) = 18t^2 + 39t + 20}$$

t = # of days of rain
$P(U(t))$ = profit.

Come up with a new name for your function. You should *not* use P or U.

Check your answer by using your new function to calculate the profit when it rains 12 days this month. Does it match what you found earlier?

$$P(U(t)) = 18t^2 + 39t + 20$$
$$P(U(12)) = 18(12)^2 + 39(12) + 20$$
$$= \$3{,}080$$

Composition Notation: We used the nested notation for the previous example. The following notations are equivalent.

$$(f \circ g)(x) = f \circ g = f(g(x))$$

Use the outputs of g as the inputs of f.

Pronounced "f of g of x"

$f(x) = 3x + 2$

$f(1) = 3 \cdot 1 + 2$

$= 5$

expl 2: Let f and g be defined below. Find the following.

$f(x) = 3x + 2$

$g(x) = 2x^2 - 1$

Remember $f(x)$ and $g(x)$ are rules. They tell you what to do to the x value.

a.) Find $(f \circ g)(4)$.

$(f \circ g)(4) = f(g(4))$

But what is $g(4)$?

$g(x) = 2x^2 - 1$

$g(4) = 2 \cdot 4^2 - 1$

$g(4) = 2 \cdot 16 - 1$

$= 32 - 1$

$= 31$

$= f(\underline{g(4)})$

$= f(\underline{31}) = 3 \cdot \underline{31} + 2 = 95$

b.) Find $(f \circ f)(1)$.

$= f(\underline{f(1)}) = f(\underline{5}) = 3 \cdot \underline{5} + 2 = 17$

c.) Find $(f \circ g)(x)$.

$= f(g(x))$

$= f(\underline{2x^2 - 1})$

$= 3(2x^2 - 1) + 2$

$(f \circ g)(x) = 6x^2 - 1$

d.) Find $(g \circ f)(x)$. $= g(f(x))$

$= g(3x + 2)$

$= 2(3x + 2)^2 - 1$

$= 2(3x + 2)(3x + 2) - 1$

$= 2(9x^2 + 12x + 4) - 1$

$= 18x^2 + 24x + 7$

Domain of Composed Functions: The domain of $f(g(x))$ is all real numbers x that are in the domain of g **and** such that $g(x)$ is in the domain of f.

expl 3: Find $f(g(x))$ and determine its domain.

$f(x) = \dfrac{1}{x - 3}$ dom: $\{x \mid x \neq 3\}$

$g(x) = \sqrt{x}$ dom: $[0, \infty)$

To find the domain of $f(g(x))$: What is the domain of $g(x)$? What is the domain of $f(x)$? Which x values would create $g(x)$ values that could **not** be inputted into $f(x)$ [i.e. not in its domain]?

domain of $f(g(x))$:

Start with dom of g, $[0, \infty)$.

But when $x = 9$, $g(9) = \sqrt{9} = 3$

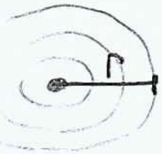
and that can't go into f.

So, dom of $f(g(x))$ is $[0, \infty)$ but need to exclude 9 too

OR $[0, 9) \cup (9, \infty)$.

Alternative method: Find the composed function and determine its domain. Then, for good measure, *also exclude* those values that do *not* work in the interior function.

$f(g(x)) = f(\sqrt{x}) = \dfrac{1}{\sqrt{x} - 3}$



r = radius

expl 4: Bob throws a rock into a still lake. The rock causes a circular ripple that gets bigger and bigger. The radius of the ripple is increasing at the rate of 2.6 feet per second. Answer the questions that follow.

a.) Find the function for the radius of the circular ripple at time t.

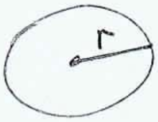
t	r
0	0
1	2.6
2	5.2
3	7.8

r = radius (ft)
t = time (sec)

$$r = 2.6t$$
or
$$r(t) = 2.6t$$

What is the radius after 0 seconds? 1 second? 2 seconds? t seconds?

b.) Find the function for the area of the circular ripple with respect to the radius r.



$$A = \pi r^2$$

$$\boxed{A(r) = \pi r^2}$$

Do you remember the area of a circle? Label it as $A(r)$.

c.) Find the composed function $(A \circ r)(t)$. What, in words, does this tell us about the ripple?

$$(A \circ r)(t) = A(r(t))$$
$$= A(2.6t)$$
$$= \pi(2.6t)^2$$
$$(A \circ r)(t) = 6.76\pi t^2$$

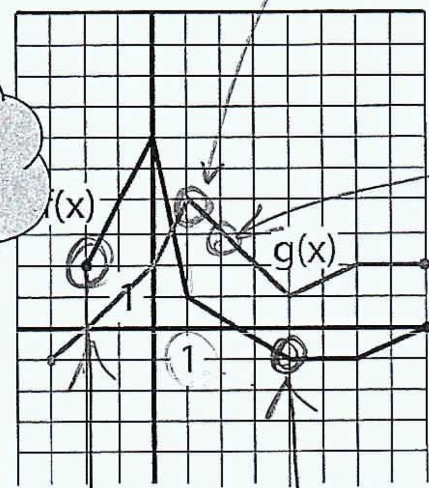
This is area of the ripple in terms of time after rock lands.

Worksheet: The Composition of Functions:

This worksheet provides practice in performing compositions. There is an application that will help explain why composition is so useful.

expl 5: Use the graphs to the right to find the following.

$g(1) = 4$



$g(2) = 3$

Start on the inside. Find $g(1)$ on the graph…

a.) $f(g(1))$

$= f(4)$

$= -1$

b.) $g(f(-2))$

$= g(2)$

$= 3$

$f(4) = -1$

$f(-2) = 2$

expl 6: Given the function $h(x)$ below, find two functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$.

$h(x) = \sqrt{4x^2 - 3}$

There are many answers.

$\boxed{\text{start } x} \rightarrow \boxed{\text{square it}} \rightarrow \boxed{\text{mult by } 4} \rightarrow \boxed{\text{subtract } 3} \rightarrow \boxed{\text{square root it}}$

$h(x)$

$g(x) = 4x^2 - 3$

$f(x) = \sqrt{x}$

Check: $f(g(x)) = f(4x^2-3) = \sqrt{4x^2-3} = h(x)$ ✓

expl 7: Let $f(x) = 3x + 2$ and $g(x) = \dfrac{x-2}{3}$. Find $f(g(x))$ and $g(f(x))$.

$f(g(x)) = f\left(\dfrac{x-2}{3}\right) = \dfrac{\cancel{3}}{1}\left(\dfrac{x-2}{\cancel{3}}\right) + 2 = x - 2 + 2 = x$

$g(f(x)) = g(3x+2) = \dfrac{3x + \cancel{2} - \cancel{2}}{3} = \dfrac{\cancel{3}x}{\cancel{3}} = x$

Why do you think these are both equal to x?