A function turns an x into a y value. How do we reverse that?

College Algebra
Class Notes
Inverses and One-to-One Functions (section 6.2)

Main Idea: Consider the function given below. What does it do to each x value to get its corresponding y value? How would we reverse the process? In other words, what would we do to a y value to get its x value back? That is what a function's inverse does for us.

Consider the function $y = 3x + 4$. Below is a verbal model that shows the relationship between x and y.

Start x → Multiply by 3 → Add 4 → End y

The following table shows some x values and their corresponding y values.

x	$y = 3x + 4$
-2	$y = 3(-2) + 4 = -2$
-1	$y = 3(-1) + 4 = 1$
0	$y = 3(0) + 4 = 4$
1	$y = 3(1) + 4 = 7$
2	$y = 3(2) + 4 = 10$

Each y value is gotten by multiplying the x value by 3 and adding 4.

But what if we wanted to go the other way? What would you do to the (y value of) 10 to get back to (the x value of) 2?

We want to reverse this process. If the original process multiplies by 3 and adds 4, how would you reverse that? Does the order of your operations matter?

$$\frac{y-4}{3} = x \qquad \frac{10-4}{3} = 2 \checkmark$$

Need to subtract 4, divide by 3.

Write down a formula for your inverse.

$$x = \frac{y-4}{3}$$

An inverse **undoes** what the original function did.

6.2

Variables of Inverses: Let's think about our use of x and y. We denote inputs as x and outputs as y. This is true for the original $y = 3x + 4$. However, when we think about its inverse, we essentially switch the roles of x and y, using y values as inputs and x values as outputs. This idea with the previous discussion leads to a general scheme for algebraically finding inverses.

Algebraic Steps for Finding Inverses:
To find the inverse of a function,

1. Switch the x and y in the equation, and
2. Solve the equation for y.

An equation where y is *not* isolated is said to be implicitly defined.

Try this procedure with $y = 3x + 4$.

Orig eqn: $y = 3x+4$
Inverse: $x = 3y+4$
$x-4 = 3y$

$y = \dfrac{x-4}{3}$

Notation: If we let $f(x)$ be a function, then we can call its inverse $f^{-1}(x)$. This is pronounced simply "f inverse of x". For instance, we could write $f(x) = 3x + 4$ and $f^{-1}(x) = \dfrac{x-4}{3}$.

This is *not* an exponent.

expl 1: Complete the following to investigate inverses.

a.) Use the function $f(x) = 3x + 4$ to find $f(12)$.

$f(12) = 3 \cdot 12 + 4 = 36 + 4 = 40$

So, $f(12) = 40$

b.) Use the function $f^{-1}(x) = \dfrac{x-4}{3}$ to find $f^{-1}(40)$.

$f^{-1}(40) = \dfrac{40-4}{3} = \dfrac{36}{3} = 12$

So, $f^{-1}(40) = 12$

c.) Explain the connection between $f(12)$ and $f^{-1}(40)$.

The original fnc took the input of 12 and outputted 40.
The inverse fnc undid this – turning 40 back into 12.

expl 2: Find the inverse of each relation.

a.) $\{ (3, 2), (-5, 4), (1, 6) \}$

b.) $x^3 y = 15$

inverse: $\{(2,3), (4,-5), (6,1)\}$

Inverse:

$y^3 x = 15$

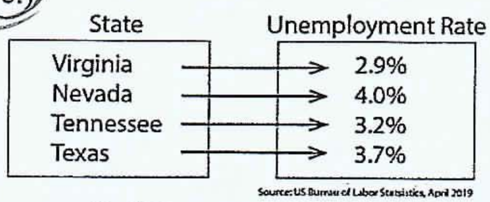
$\frac{y^3 \cancel{x}}{\cancel{x}} = \frac{15}{x}$

$y^3 = 15/x$

$\sqrt[3]{y^3} = \sqrt[3]{15/x}$

$\boxed{y = \sqrt[3]{15/x}}$ inverse

c.)

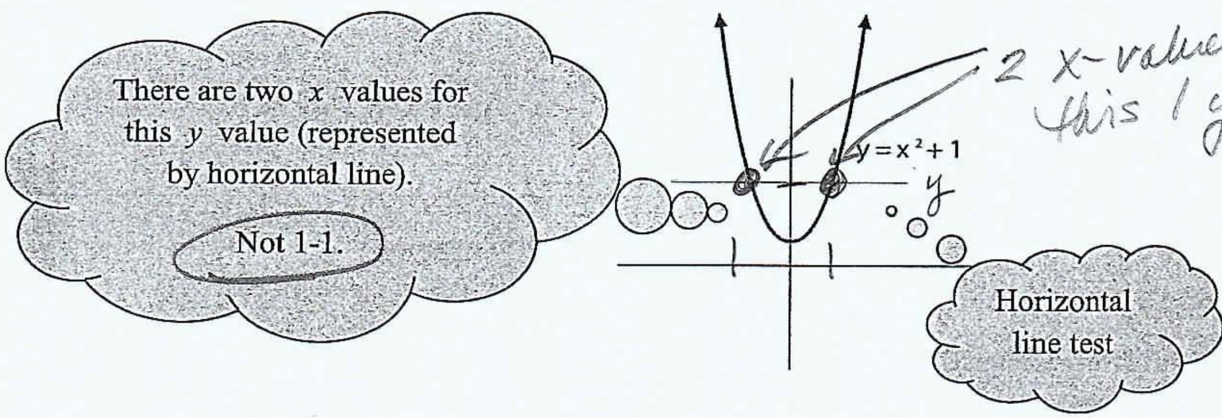


2.9% → VA
4.0% → NV
3.2% → TN
3.7% → TX

defn fnc: for every x value, there's exactly 1 y value

Definition: One-to-One Functions:

A function is **one-to-one (1-1)** if for every y value, there is exactly one x value. Consider the function below.



2 x-values for this 1 y-value.

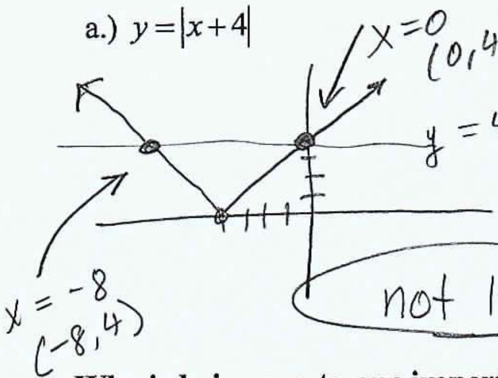
A function is 1-1 if different inputs have different outputs. Or rather, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. Another way to state this is, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. You would use this to prove a function is 1-1 or to algebraically show it is *not*.

(not crucial for us.

older calculators: $y_1 = abs(x+4)$

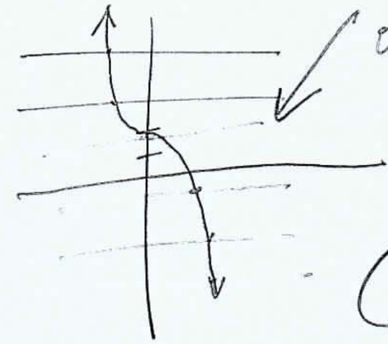
expl 3: Graph the function and determine if it is one-to-one. Copy your graphs here.

a.) $y = |x+4|$



not 1-1

b.) $g(x) = -x^3 + 2$



only 1 x-value for every y-value.

1-1

Why is being one-to-one important?

Remember how $y = x^2 + 1$ was found to *not* be one-to-one? Find the inverse of $y = x^2 + 1$. Is this inverse a function?

orig fnc: $y = x^2 + 1$

inverse: $x = y^2 + 1$

$x - 1 = y^2$

$\sqrt{x-1} = \sqrt{y^2}$

inverse

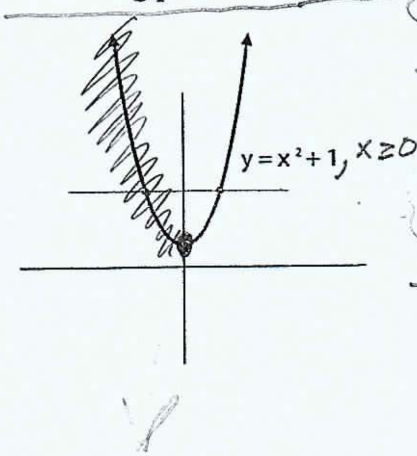
$\boxed{y = \pm\sqrt{x-1}}$

Is this a fnc?
(Put in $x=5$)
$\rightarrow y = \pm\sqrt{5-1} = \pm\sqrt{4} = \pm 2$

not a fnc

Restricted Domains: It turns out that if a function is *not* 1-1, then its inverse will *not* be a function. And that is a problem because algebra loves functions. What's a person to do?

We could restrict the domain of the original function so that its inverse is a function. Consider the function below whose domain is currently "all real numbers". Lop off part of it so that the remaining part is one-to-one. What is the new (restricted) domain?



(new) restricted domain:

$[0, \infty)$ or $x \geq 0$

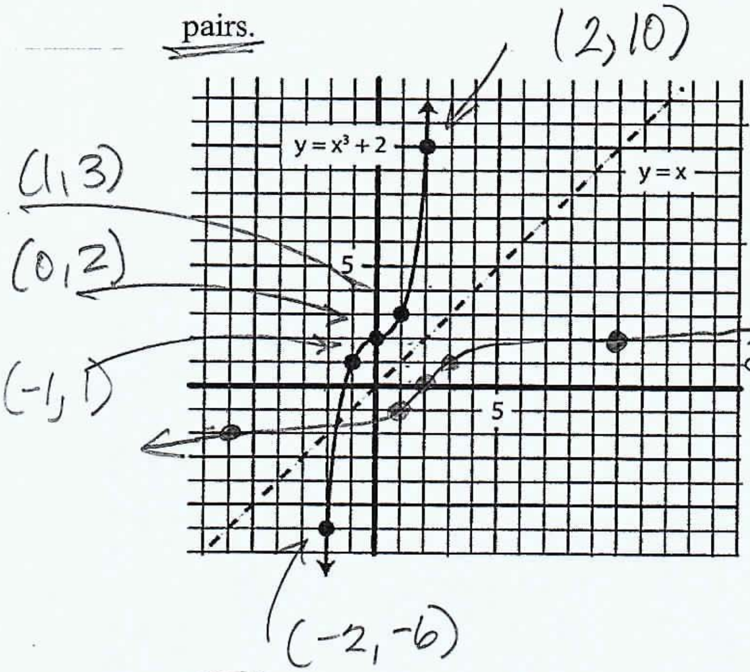
What is the inverse of this new function, with its restricted domain?

inverse: $y = \pm\sqrt{x-1}$ But we need $y \geq 0$.

So, really inverse is $y = \sqrt{x-1}$

Graphical Interpretation of Inverses:

expl 4: Find the inverse for the function $y = x^3 + 2$ pictured below by reversing the ordered pairs.



Reversing the points' coordinates is akin to switching x and y to find the inverse.

How are the graphs of a function and its inverse related?

mirror images of each other over the line $y=x$.

expl 5: Consider the function $g(x) = \sqrt{x+3}$. Answer the following questions.

a.) Algebraically find the inverse of g.

orig fnc: $y = \sqrt{x+3}$

inverse: $x = \sqrt{y+3}$

$x^2 = (\sqrt{y+3})^2$

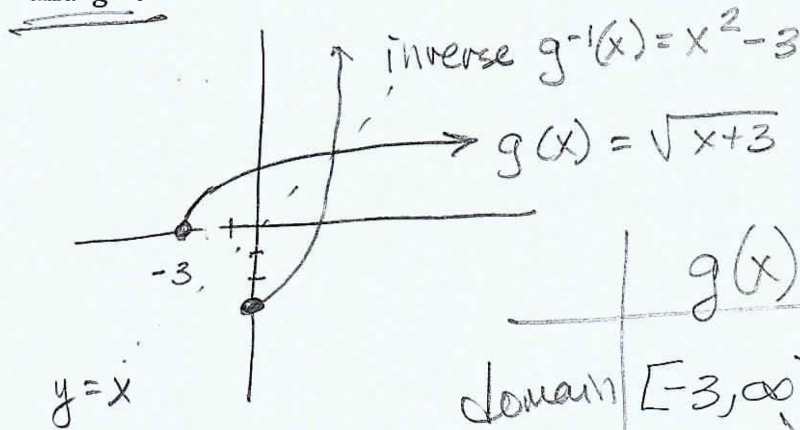
$x^2 = y+3$

$x^2 - 3 = y$

$\boxed{y = x^2 - 3}$ inverse

Consider the graphs to check your inverse.

b.) Graph both functions on the same plane. Determine the domain and range for both g and g^{-1}.



How are the domains and ranges related?

	$g(x)$	$g^{-1}(x)$
domain	$[-3, \infty)$	$[0, \infty)$
range	$[0, \infty)$	$[-3, \infty)$

Graphing functions with restricted domains: Your calculator will graph with restricted domains. For instance, let's graph the function and its inverse from the previous example.

Enter the following into the calculator.

$Y_1 = \sqrt{\ }(x+3)$
$Y_2 = (x^2-3)(x \ge 0)$

We would write this as $y = x^2 - 3, x \ge 0$ on paper.

The inequality symbol is found under TEST which is the second function of MATH.

Use the Zoom: ZSquare setting.

expl 6: The formula $C(f) = \frac{5}{9}(f-32)$ converts Fahrenheit temperatures f to Celsius temperatures $C(f)$.

a.) Find $C(95)$.

$C(95) = \frac{5}{9}(95-32) = \boxed{35°C}$

To denote the inverse of a function that does *not* use x and y, the inverse here may be written as $f(C)$ instead of $C^{-1}(f)$.

b.) Define f and C for the function $C(f)$ to explain the answer in part *a*.

So, $95°F = 35°C$

An input of f gives an output of C.

c.) Find the inverse $f(C)$ and explain what it represents.

We do *not* "switch x and y". Rather, solve for f and finish using function notation.

orig fnc: $C = \frac{5}{9}(f-32)$

$\frac{9}{5}C = \cancel{\frac{9}{5}}\cancel{\frac{5}{9}}(f-32)$

$\frac{9}{5}C = f-32$

$\frac{9}{5}C + 32 = f$ → $f = \frac{9}{5}C + 32$ or $\boxed{f(C) = \frac{9}{5}C + 32}$

d.) Without doing the calculation, what do you think $f(35)$ is equal to? Explain.

$f(35) = 95°F$

The fnc f(c) undoes what C(f) does. (and C(95) = 35 from part a.)

expl 7: For the function f, use composition to show that g is its inverse. In other words, show that $f(g(x)) = x$ and $g(f(x)) = x$.

$f(x) = \frac{2}{5}x + 1$

$g(x) = \frac{5x-5}{2}$

Can you see how this shows each function undoes the other?

You may need to review composition.

→ Show $f(g(x)) = x$.

Find $f(g(x)) = f\left(\frac{5x-5}{2}\right)$

$= \frac{\cancel{2}}{5}\left(\frac{5x-5}{\cancel{2}}\right) + 1$

$= \frac{\cancel{5}(x-1)}{\cancel{5}} + 1$

Do we need to exclude values from the domains of f or g?

$= x - 1 + 1$

$= x$ ✓ So, $f(g(x)) = x$.

Show $g(f(x)) = x$.

Find $g(f(x)) = g\left(\frac{2}{5}x + 1\right)$

$= \frac{\cancel{5}\left(\frac{2}{\cancel{5}}x + 1\right) - 5}{2}$

$= \frac{2x \cancel{+5} \cancel{-5}}{2}$

$= \frac{\cancel{2}x}{\cancel{2}}$

$= x$ ✓, So, $g(f(x)) = x$.

Since $f(g(x)) = x$ and $g(f(x)) = x$, we see f and g are inverses.