We have studied many types of functions and their graphs. Here is a new class of function.

College algebra
Class notes
Exponential Functions and Their Graphs (section 6.3)

Definition: Exponential Function:

An **exponential function** is of the form $f(x) = a^x$ where a is a positive real number *not* equal to 1. The number a is called the **base** or **growth factor**. Notice the variable x is in the exponent position. This is what makes it an exponential function.

$y = 2^x$
or $y = 5^x$
or $y = 100^x$

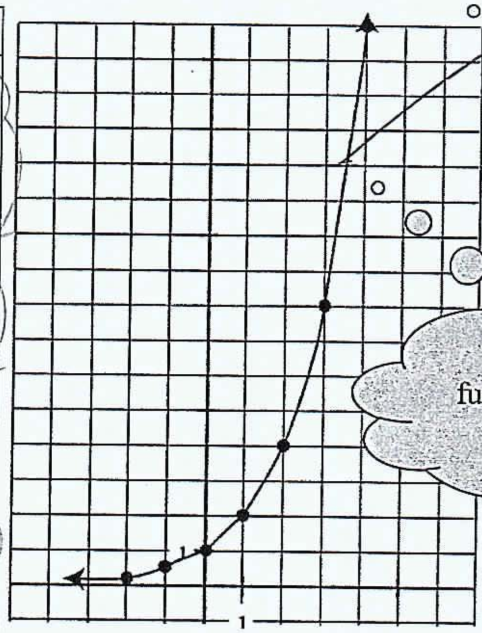
We will also see functions in the form $f(x) = Ca^x$. Here, C is a real number *not* equal to 0 and it is called the **initial value**. Do you see why?

$f(0) = C \cdot a^0 = C$ — y value when $x = 0$.

Domain: Since you could raise a positive number to any power and get a real number out, there are no real numbers we need to exclude from the domain. So, **the domain of any exponential function is all real numbers.**

expl 1: Consider $y = 2^x$. Inspect the table and graph.

x	$y = 2^x$
-2	$2^{-2} = \frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$



increasing

Compare this table and graph to that of a linear function.

All exponential functions will have this general shape.

I call the shape "shloopy".

$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

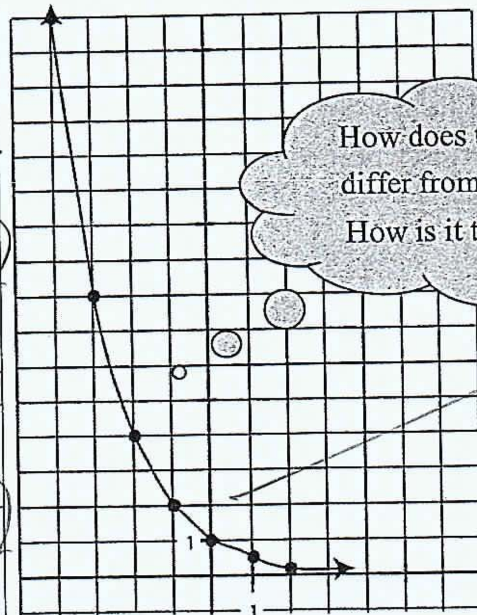
$a^{-n} = \frac{1}{a^n}$ $(a \neq 0)$

Exponent Rule:

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$$

expl 2: Consider $y = \left(\frac{1}{2}\right)^x$. Inspect the table and graph. $\left(\frac{1}{2}\right)^{-4} = \left(\frac{2}{1}\right)^4 = 2^4 = 16$

x	$y = \left(\frac{1}{2}\right)^x$
-4	$\left(\frac{1}{2}\right)^{-4} = 2^4 = 16$
-3	$\left(\frac{1}{2}\right)^{-3} = 2^3 = 8$
-2	$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$
-1	$\left(\frac{1}{2}\right)^{-1} = 2^1 = 2$
0	$\left(\frac{1}{2}\right)^{0} = 1$
1	$\left(\frac{1}{2}\right)^{1} = \frac{1}{2}$
2	$\left(\frac{1}{2}\right)^{2} = \frac{1}{4}$



How does this graph differ from $y = 2^x$? How is it the same?

decreasing

What about the function makes the difference?

y values halve as we go down table

Why does $y = 2^x$ increase and $y = \left(\frac{1}{2}\right)^x$ decrease? What do you think makes the difference?

Predict whether the following functions would increase or decrease.

a.) $y = 4^x$ increase

b.) $y = 13^x$ increase

c.) $y = \left(\frac{1}{3}\right)^x$ decrease

↳ y values in table would decrease by $\frac{1}{3}$ each time

d.) $y = .67^x$ decrease (base < 1)

e.) $y = \left(\frac{5}{4}\right)^x$ increase (base > 1)

For $y = a^x$, it will increase if $a > 1$ and decrease if $a < 1$.

Definition: Natural Exponential Function:
This is a special exponential function whose base is e, an irrational number (meaning its decimal form does *not* terminate or repeat) approximately equal to 2.71828. The **natural exponential function** is $f(x) = e^x$.

expl 3: Find the following ~~using a~~ calculator. Round to four decimal places.

a.) e^{-5} ≈ 0.0067

> Calculator: You will see e^x is the second function of the LN button on the left side.

b.) $\left(\frac{1}{e^3}\right)^2$ ≈ 0.0025

Optional Worksheet: Working with Exponential Functions
This worksheet will explore the two shapes we see with the graphs of exponential functions. It will also review simplifying exponents and using function notation. Solutions are available online.

Review of Exponent Rules: You will use many of these rules as you manipulate expressions.

Product rule: $a^m \cdot a^n = a^{m+n}$

Quotient rule: $\frac{a^m}{a^n} = a^{m-n}$

Power rule: $\left(a^m\right)^n = a^{m \cdot n}$

Power of a product rule: $(a \cdot b)^n = a^n \cdot b^n$

Power of a quotient rule: $\left(\frac{a}{c}\right)^n = \frac{a^n}{c^n}$

Zero exponent rule: $a^0 = 1$ (Here a cannot be 0 because 0^0 is undefined.)

Negative exponent rule: $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$ (if a is non-zero and n is an integer).

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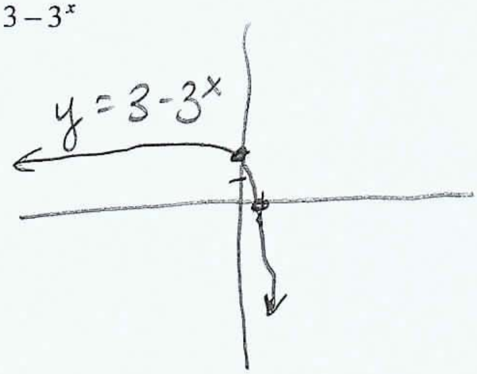
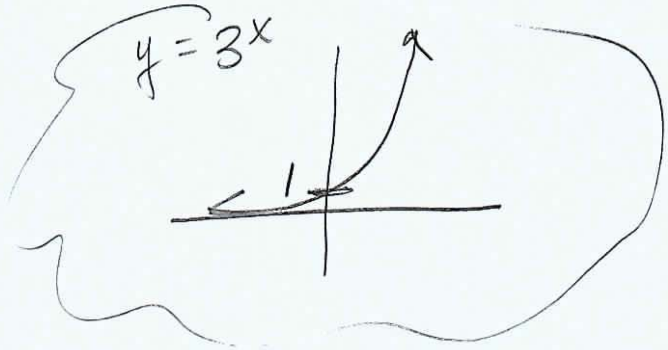
expl 4: Graph this function on the calculator. Use the standard window. This graph can be thought of as a transformation. Can you see the mother function $y = 3^x$ in this graph?

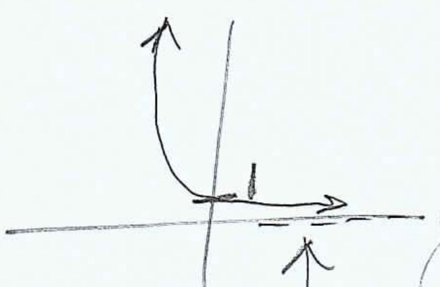
$y = 3 - 3^x$





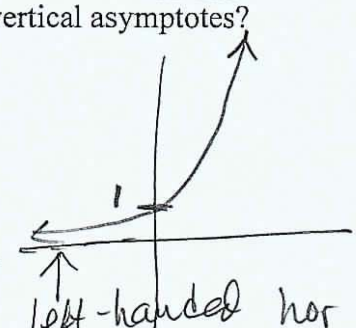
We can use transformations to help graph these functions by hand. Recall the general shape of $f(x) = a^x$ when $0 < a < 1$ and when $a > 1$. Draw them right now below. What is the y-intercept of each graph? Do they have horizontal or vertical asymptotes?

$y = a^x,\ a > 1$



$y = a^x,$
$0 < a < 1$

$y = \left(\frac{1}{2}\right)^x$



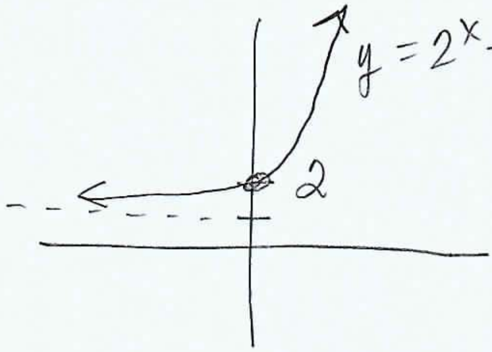
$y = 2^x$

left-handed hor asym $y = 0$

right-hand hor asymp $y = 0$

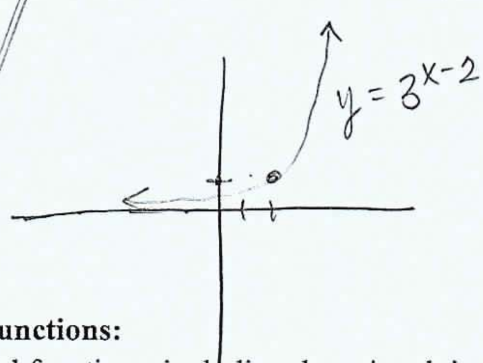
expl 5: Use transformations to sketch the graph of the following functions. Check on your calculator.

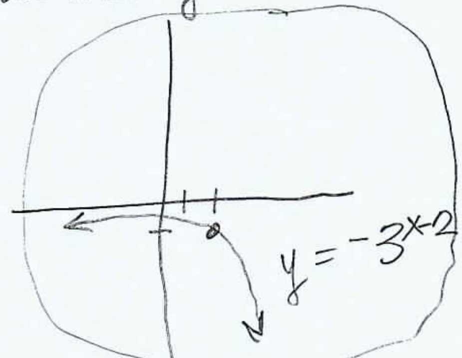
a.) $y = 2^x + 1$



b.) $f(x) = -3^{x-2}$

mother func : $y = 3^x$
trans #1 $y = 3^{x-2}$
trans #2 $y = -3^{x-2}$





Worksheet: Exploring Exponential Functions:

We will look at the graphs of exponential functions, including those involving transformations. We will also practice finding exponential values using the calculator.

t = time (months)
V = value of stock ($)

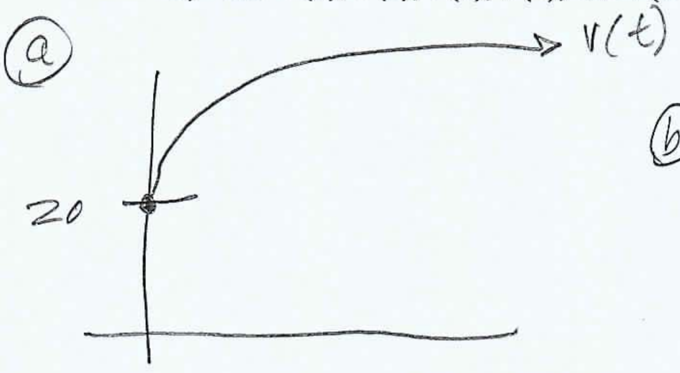
expl 6: The value of a stock is given by the function $V(t) = 58\left(1 - e^{-1.1t}\right) + 20$ where V is the value of the stock after time t, in months.

a.) Graph the function. Copy it here.

b.) Find $V(1)$, $V(2)$, $V(4)$, $V(6)$, and $V(12)$. Interpret these values.

Can you use the VALUE function under CALCULATE?

(a)



$[0, 15] \times [0, 100]$

(b) $V(1) \approx \$58.69$

$V(2) \approx \$71.57$

$V(4) \approx \$77.29$

$V(6) \approx \$77.92$

$V(12) \approx \$78.00$

Solving Special Exponential Equations:

What would you say the solution to $5^x = 5^4$ is? Go with your gut. There is actually a pretty simple property that backs up what your gut probably told you.

$x = 4$

Base-Exponent Property: For any $a > 0$ and $a \neq 1$, we know that $a^x = a^y$ if and only if $x = y$.

[defn of exp fnc.

So if $3^x = 3^7$, then what must be true?

Let's look at this in action, but we may have to work to get our equations in this perfect form.

✓ pg 3 Exponent Rules

expl 7: Solve the following equations.

a.) $\left(\frac{1}{4}\right)^x = \frac{1}{64}$

$4^3 = 64$

$\left(\frac{1}{4}\right)^x = \left(\frac{1}{4}\right)^3$

$x = 3$

Check: $\left(\frac{1}{4}\right)^3 \stackrel{?}{=} \frac{1}{64}$

calc ✓ calc.

b.) $4^x \cdot 2^{x^2} = 16^2$

$4 = 2^2$
$16 = 2^4$

$(2^2)^x \cdot 2^{x^2} = (2^4)^2$

$2^{2x} \cdot 2^{x^2} = 2^8$

$2^{2x + x^2} = 2^8$ ★ Base-Exp Property Form

$2x + x^2 = 8$

$x^2 + 2x - 8 = 0$

✓ $(x + 4)(x - 2) = 0$

$x + 4 = 0$ $x = -4$ or $x - 2 = 0$ $x = 2$

Check

$$4^x \cdot 2^{x^2} = 16^2$$

Check: $x = -4$ ✓

$$\underbrace{4^{-4} \cdot 2^{(-4)^2}}_{\text{calc}} \overset{?}{=} \underbrace{16^2}_{\text{calc.}}$$

✓

$x = 2$

$$\underbrace{4^2 \cdot 2^{2^2}}_{\text{calc.}} \overset{?}{=} \underbrace{16^2}_{\text{calc.}}$$

✓