College algebra
Class notes
Logarithmic Functions and Their Graphs (section 6.4)

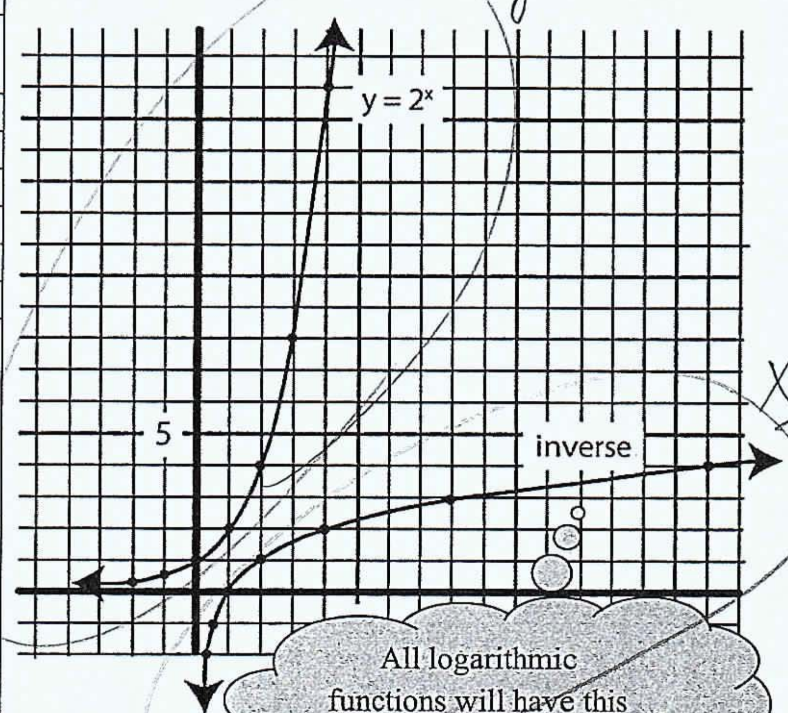
We'll use this new name for the *inverse* of the exponential function.

Let's investigate the inverse of the exponential function from the previous section. I have recreated the table of values and graphed the exponential function $y = 2^x$ below.

Then I switched the x and y values in the equation and table. I graphed the resulting inverse relation using the points from the table. Is this inverse a function?

$y > 0$

Exponential		Inverse	
x	$y = 2^x$	x	y in $x = 2^y$
-2	¼	¼	-2
-1	½	½	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3
4	16	16	4



$x > 0$

Is the exponential function one-to-one? If so, its inverse will be a function.

All logarithmic functions will have this general shape.

$x = 2^y \iff y = \log_2 x$

So, to find the equation for the inverse, we'd normally take the equation $x = 2^y$ and solve for y. But we do *not* know how to isolate y. So we'll invent new notation and write $y = \log_2 x$ to mean the same as $x = 2^y$. We need to be able to interpret this new log notation.

In words, how would you describe y in the equation $x = 2^y$? Use the right-side table above if you need. In other words, how is y related to 2 and x?

$x = 2^y$ y is exponent for the base of 2 that will make x

(y is the exponent I raise 2 to, to get x.)

Meaning of Logarithms:

We will use the idea from the bottom of page 1 to define what $\log_2 x$ means. We will say that "$\log_2 x$ is the number to which I raise 2 to get x". This is very important in our study of logs.

expl 1) Use the fact that "$\log_a x$ is the number to which I raise a to get x" to figure the following logs without the calculator.

$3^{?} = 27$

a.) $\log_3 27$ = 3 because $3^3 = 27$.

"$\log_3 27$ is the number to which I raise 3 to get 27"

b.) $\log_5\left(\dfrac{1}{125}\right)$ = the # to which I raise 5 to get $\frac{1}{125}$

So $\log_5\left(\frac{1}{125}\right) = -3$

$5^3 = 125$

$\frac{1}{5^3} = \frac{1}{125}$

$5^{-3} = \frac{1}{125}$

c.) $\log_{10}\sqrt{10}$ = the # to which I raise 10 to get $\sqrt{10}$.

$\log_{10}\sqrt{10} = \log_{10} 10^{1/2} = 1/2$

$\sqrt{10} = 10^{1/2}$

$\sqrt{a} = a^{1/2}$

$\sqrt[n]{a} = a^{1/n}$

d.) $\log_6 6^3$

This last one will lead us to a log rule in the next section.

$6^{?} = 6^3$

the # to which I raise 6 to get 6^3 $\rightarrow \log_6 6^3 = 3$

Definition: Logarithmic Function:

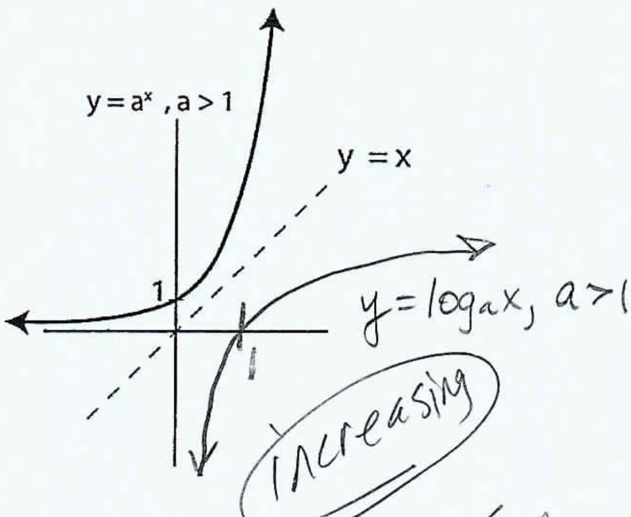
We define $y = \log_a x$ to be the number y such that $x = a^y$. Because of its connection to the exponential relationship, we say $x > 0$ (this is the domain of the function) and a is a positive constant *not* equal to 1.

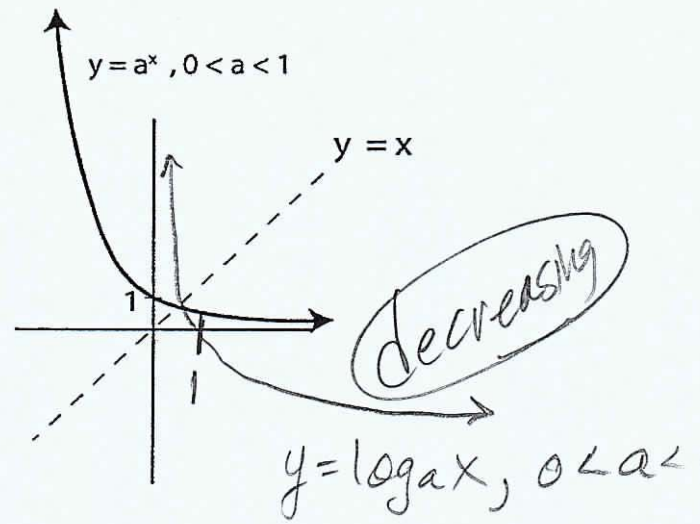
A more useful way to define logs, as stated above, is $\log_a x$ is the number to which I raise a to get x.

Sometimes written as $y = \log_a(x)$.

The number a is called the **base**. Notice it is the same as the base of the exponential function from which it came.

Graphs: The graphs of logarithmic functions will come in two different flavors, just like exponential graphs. Below are the graphs of the basic exponential functions. Reflect them over the line $y = x$ to get their logarithmic inverses.





Are these logarithmic functions one-to-one? What are their domains? What are their ranges? What are their x and y-intercepts? Are they increasing or decreasing?

yes

x-int : $x = 1$

no y-int

$x > 0$

all real numbers

The characteristics of *transformed* log functions will change accordingly. What's the domain of $y = \log_2(x-5)$?

Definition: Common Logarithmic Function:

If 10 is the base of the logarithm, we have $y = \log_{10} x$. We will call this the **common logarithmic function**. We can abbreviate "$\log_{10}$" as simply "log" with no base apparent.

Definition: Natural Logarithmic Function:

If e is the base of the logarithm, we have $y = \log_e x$. We will call this the **natural logarithmic function**. We can abbreviate "$\log_e$" as "ln".

natural exp. fnc $y = e^x$

$y = \ln(x)$

Calculator usage:
You will see two buttons on your calculator, LN and LOG. These are base e and base 10 logs. To find logs of other bases, we will probably need a change-of-base formula discussed later.

expl 2: Find each using the calculator. Round to three decimal places.

a.) log 650 $= \log_{10} 650 \approx 2.813$ because $10^{2.813} = 650$

b.) ln 80.56 $= \log_e 80.56 \approx 4.389$ because $e^{4.389} = 80.56$

c.) $\dfrac{\ln \frac{4}{3}}{0.06} \approx 4.795$

d.) $\log_{10} -20 = $ $10^{?} = -20$

Why does $\log_{10} -20$ give you an error?

dne. or non-real

Change-of-Base Formula:
In the next section, we will see a formula that allows us to find logs of bases other than 10 or e on the calculator. Some newer calculators will do this inherently but older models will *not*.

Worksheet: Visiting with exponential and logarithmic functions:
This worksheet will explore the relationship between exponential functions and their inverses, logarithmic functions. We will also work on understanding what a logarithm means.

Convert between exponential equations and logarithmic equations:

We have the general notion that $y = \log_a x$ and $x = a^y$ are equivalent. That means, if we have an equation in exponential form, we should be able to convert it to logarithmic form using these equations as a guide, and vice versa.

You can also do this conversion by thinking about how "$\log_a x$ is the number to which I raise a to get x".

expl 3: Convert the logarithmic equation to the equivalent exponential equation.

a.) $\log_{10} 10{,}000 = 4$

$\Downarrow$

$10^4 = 10{,}000$

Keep in mind
$y = \log_a x \longleftrightarrow x = a^y$

b.) $\log_3 x = 4$

$x = 3^4$ ✓

c.) $\log_x y = 0.845$

$x^{0.845} = y$

d.) $\ln 4 = x$ $\quad \log_e 4 = x$

$\Downarrow$

$e^x = 4$

This will help us solve equations later.

expl 4: Convert the exponential equation to the equivalent logarithmic equation.

a.) $4^x = 64$ $\quad \log_4 64 = x$

$x = a^y$
$\Updownarrow$
$y = \log_a x$

b.) $10^y = 9764$ $\quad \log_{10} 9764 = y$

c.) $e^6 \approx 403.4$ $\quad \log_e 403.4 \approx 6$

OR $\ln 403.4 \approx 6$

Solving Some Logarithmic Equations:

We can use the equivalence of $\boxed{y = \log_a x \text{ and } x = a^y}$ to solve certain log equations as hinted at on the last page. Let's see this in action.

expl 5: Convert to exponential form and then solve the equations for x.

a.) $\log_3 x = 4$ $\rightarrow$ $3^4 = x$

$x = 81$

The variable is buried within a log. Use the equivalence to unbury it.

b.) $\log_3 (2x+8) = 4$

$3^4 = 2x+8$

$81 = 2x+8$

$73 = 2x$

$x = 73/2$

OR

$x = 36.5$

expl 6: To solve this one, notice the x is *not* within the log and so the above trick will *not* work. Rather than converting to exponential form, determine what $\log_4 4096$ is and then continue to solve for x.

$\boxed{\log_4 4096 = 3x - 5}$

$6 = 3x - 5$ because $\log_4 4096 = 6$

(because $4^6 = 4096$)

$11 = 3x$

$11/3 = x$

$N(a) = \#$ of units sold
a = 1000's of dollars spent on ads

expl 7: A model for advertising response is given by $\boxed{N(a) = 1000 + 200 \ln a, \quad a \geq 1}$ domain. Here $N(a)$ is the number of units sold when a thousand dollars is spent on advertising.

a.) How many units would be sold if they spend \$5,000 on advertising?

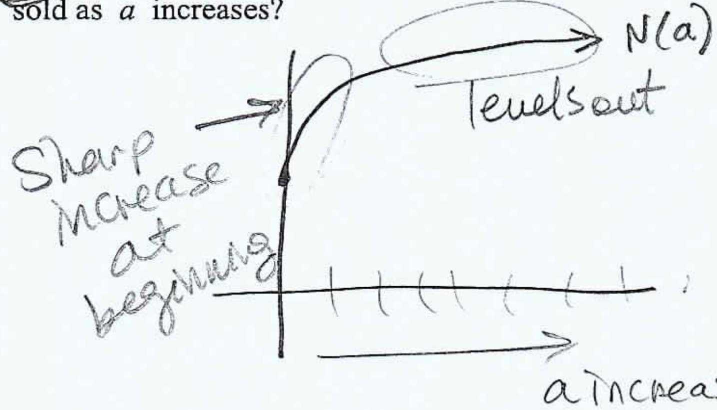
What is a?

$N(5) = ?$ $\qquad a = 5$

$N(a) = 1000 + 200 \cdot \ln a$

$N(5) = 1000 + 200 \cdot \ln 5 \approx 1322$ units sold

b.) Graph the function on the window [0, 25] x [0, 2000]. What happens to the number of units sold as a increases?



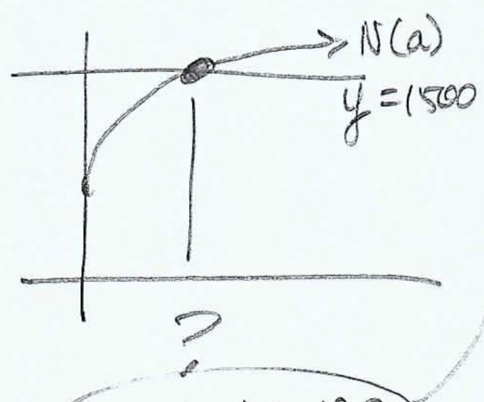
As a increases, the # of units sold increases and then levels out.

c.) The company would like to sell 1500 units. How much should be spent on advertising? Solve graphically.

$N(a) = 1500 \qquad a = ?$

$N(a) = 1000 + 200 \cdot \ln a$

Solve $1500 = 1000 + 200 \cdot \ln a$



$a \approx 12.182$

So, to sell 1500 units, they ought to spend \$12,182 on ads.