We will use our log rules and the equivalency of $x = a^y$ and $y = \log_a x$.

College algebra
Class notes
Solving Exponential and Logarithmic Equations (section 6.6)

Recall: Definition: Exponential Equation: An equation with the variable in the exponent position will be called an exponential equation.

Examples:

$5^x = 5^4$, $e^{2t} = 500$, $3^{4x} = 81$, $3^x = 2^{x-1}$, $e^x - 6e^{-x} = 1$, $27 = 3^{5x} \cdot 9^{x^2}$

Recall: Definition: Logarithmic Equation: An equation that has logs of variable expressions will be called a logarithmic (or log) equation.

Examples:

$\log_3(4w) = 4$, $3\log_2(x-1) + \log_2 4 = 5$, $\log_4 x + \log_4(x-3) = 1$, $\log_2(x-1) - \log_6(x+2) = 2$

We're solving equations in these sections. Remember we are trying to find the x that makes the equation true. We will need to keep a lot in our minds.

1.) the logarithm rules from the previous section
2.) the fact that $x = a^y$ and $y = \log_a x$ are equivalent
3.) the two properties below

Solve $5 = 2x + 1$
-1
$4 = 2x$
$\frac{4}{2} = \frac{2x}{2}$
$x = 2$

Base-Exponent Property: For any $a > 0$ and $a \neq 1$, we know that $a^x = a^y$ if and only if $x = y$.

So if $5^x = 5^4$, then what must be true?

Logarithmic Equality Property: For any $M > 0$, $N > 0$, $a > 0$, and $a \neq 1$, we know that $M = N$ if and only if $\log_a M = \log_a N$.

So if $\log_2 x = \log_2 10$, then what must be true?

Many of the problems can be solved by various methods. We will explore this with the examples. When doing homework, choose the method that most appeals to you or that best fits that equation.

Solving Exponential Equations:

Change of Base

$\log_4 20 = \frac{\log_{10} 20}{\log_{10} 4}$

expl 1: Solve. Try the different methods below.

$4^{2t+1} = 20$

Method 1: Use the equivalency of $x = a^y$ and $y = \log_a x$ to rewrite the equation in log form.

$\log_4 20 = 2t+1$

$2.16 \approx 2t+1$

$-1 \qquad -1$

$1.16 \approx 2t$

$\frac{1.16}{2} \approx \frac{2t}{2}$

$0.58 \approx t$ ✓

You're *not* done until t is isolated.

Check:

$4^{(2t+1)} = 20$

$4^{2(0.58)+1} \stackrel{?}{=} 20$

calc. ✓

Method 2: Use the **Logarithmic Equality Property** to "take the log (base 4) of both sides".

$4^{2t+1} = 20$

$\log_4 4^{2t+1} = \log_4 20$

Rule 6

$2t+1 = \log_4 20$

$\vdots$

$t \approx 0.58$

$\ln 4^{2t+1}$

Rule 3

$= \log_e 4^{2t+1}$

$= (2t+1) \cdot \log_e 4$

$= (2t+1) \ln 4$

Method 3: Use the **Logarithmic Equality Property** to "take the natural log of both sides".

$4^{2t+1} = 20$

$\ln 4^{2t+1} = \ln 20$

$\frac{(2t+1)\cdot \cancel{\ln 4}}{\cancel{\ln 4}} = \frac{\ln 20}{\ln 4}$

$2t+1 \approx 2.16$

$t \approx 0.58$

Which method do you like best?

If you know your powers of 3, it might get you going here.

expl 2: Solve.

$3^{4x} = 81$

$3^{4x} = 3^4$

$4x = 4$

$x = 1$

This looks crazy but substitute y for 2^x and …

expl 3: Solve.

$2^{2x} + 2^x - 12 = 0$

Let $y = 2^x$,

then our eqn becomes

$y^2 + y - 12 = 0$

$(y+4)(y-3) = 0$

$y + 4 = 0$ or $y - 3 = 0$

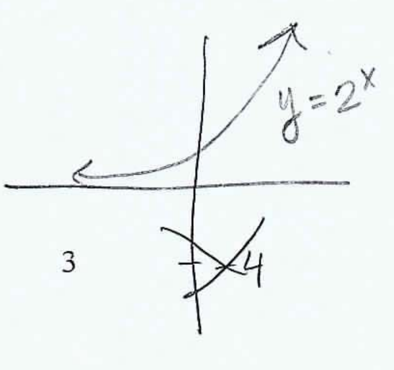
$y = -4$ or $y = 3$

So, $\cancel{2^x = -4}$ or $2^x = 3$

(Method 1 from pg 2)

$\log_2 3 = x$

$x \approx 1.58$ ✓



We know…

$2^{2x} = (2^x)^2$

exp rule: $(a^n)^m = a^{n \cdot m}$

Check $x = 1.58$

$\underbrace{2^{2x} + 2^x - 12}_{\text{calc}} \stackrel{?}{=} 0$ ✓

Always check your answers.

Solving Logarithmic Equations: Some equations will need to be simplified using our newly learned log rules in addition to using the exponential and logarithmic properties on page 1.

expl 4: Solve. Try the different methods below.

$\log_3(4w)=4$

Method 1: Use the equivalency of $x=a^y$ and $y=\log_a x$ to rewrite the equation in exponential form.

$3^4 = 4w$

$81 = 4w$

$\frac{81}{4} = \frac{\cancel{4}w}{\cancel{4}}$

$20.25 = w$

Check:

$\log_3(4 * 20.25) \stackrel{?}{=} 4$

$\log_3(81) \stackrel{?}{=} 4$

✓

(because $3^4=81$)

Always check your answers.

Method 2: Use the **Base-Exponent Property** to rewrite this as an exponential equation. Then use the log rules to simplify as you solve for w.

What base will you use?

$\log_3(4w)=4$

if $x=y$, then $a^x=a^y$

$3^{\log_3(4w)} = 3^4$

Rule 7

$4w = 81$

$w = 20.25$

Check: $x = 1+\sqrt{5}$ ✓

expl 5: Solve. Express irrational solutions in exact form.

$2\log_2(x-1) = \log_2 5$

Can we get this to the $\log_a M = \log_a N$ form?

$M = N$ (Log. Equal Prop (pg1))

(Rule3) $\log_2 (x-1)^2 = \log_2 5$

$\downarrow$

$(x-1)^2 = 5$

$\sqrt{(x-1)^2} = \sqrt{5}$

$x-1 = \pm\sqrt{5}$

$\cancel{x = 1 \pm \sqrt{5}}$

$\boxed{x = 1+\sqrt{5}}$

Check: $\cancel{x = 1-\sqrt{5}}$

$\underbrace{\cancel{2\cdot\log_2(\text{ANS}-1)}}_{\text{calc.}} \stackrel{?}{=} \log_2 5$

$\underbrace{2\cdot\log_{10}(\text{ANS}-1)/\log_{10}(2) \stackrel{?}{=} \log_{10}5/\log_{10}2}_{\text{Change of Base Formula}}$

expl 6: Solve.

Get the logs together and use your log rules. Can you rewrite this as a quadratic equation and solve?

$\log_3(2x-5) = 1 - \log_3 x$

$+\log_3 x \quad +\log_3 x$

Always check your answers in the *original* equation.

$\log_3(2x-5) + \log_3 x = 1$

Rule 2 $\log_3[(2x-5)\cdot x] = 1$

$\downarrow$ equiv exp form $y=\log_a x \Leftrightarrow a^y = x$

$3^1 = (2x-5)\cdot x$

$3 = 2x^2 - 5x$

$0 = 2x^2 - 5x - 3$

$0 = (2x+1)(x-3)$

$2x+1=0$ or $x-3=0$

$\cancel{x = -1/2}$ $\qquad$ $x=3$ ✓

Check: $\log_3(2x-5) = 1 - \log_3 x$

$x = -1/2$

$\log_3(2(-\tfrac{1}{2})-5) \stackrel{?}{=} 1 - \log_3(-1/2)$

$\cancel{\log_3(-6) \stackrel{?}{=} 1 - \log_3(-1/2)}$

The methods we use sometimes produce extraneous solutions. So you *must* check your solutions.

Check: $x=3$

$\log_3(2\cdot 3-5) \stackrel{?}{=} 1 - \log_3(3)$

$\log_3(1) = 1-1$

Solving Equations Graphically: As we have seen before, solving an equation graphically is simply a matter of graphing "$y=$ the left side" and "$y=$ the right side" and seeing where they intersect. One advantage of a graphical solution is that you *never* get extraneous solutions.

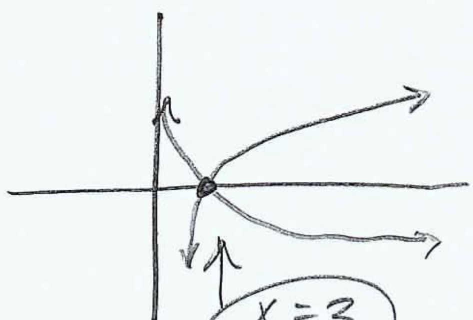
expl 7: Solve using a graphing calculator. Copy the graph here. Do *not* just TRACE. Use the INTERSECT function on the calculator. Round your solutions to three decimal places.

$\log_3(2x-5)=1-\log_3 x$

You will need the Change of Base Formula. Do *not* use the other log rules.

$y_1 = \log(2x-5)/\log(3)$

$y_2 = 1-\log(x)/\log(3)$



$x=3$

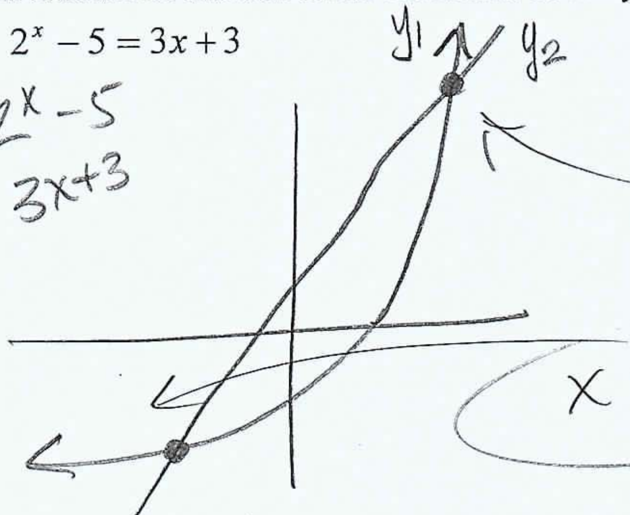
expl 8: Solve using a graphing calculator. Copy the graph here. Do *not* just TRACE. Use the INTERSECT function on the calculator. Round your solutions to three decimal places.

$2^x-5=3x+3$

Did you find both solutions?

$y_1=2^x-5$

$y_2=3x+3$



$x \approx 4.408$

$x \approx -2.612$

$[-10,10] \times [-10,20]$

Worksheet: Using log rules to solve equations:

This ~~worksheet guides you through solutions~~ with step-by-step instructions, providing practice solving equations both algebraically and graphically. It gives good advice on how to graph the pieces of these equations.

expl 9: The value of a Chevy Cruze LT that is t years old can be modeled by $V(t) = 19{,}200(0.82)^t$. Answer the following questions.

$V(t)$ = value of Chevy Cruze ($)

t = age (years)

a.) How much is the car worth when $t = 0$? Interpret this result.

$$V(0) = 19{,}200\,(\cancel{0.82})^{0} \to 1$$

$$\boxed{V(0) = \$19{,}200}$$ — Value of car when it's brand new.

b.) When will the car be worth $10,000? Round to the nearest tenth of a year.

Solve this graphically.

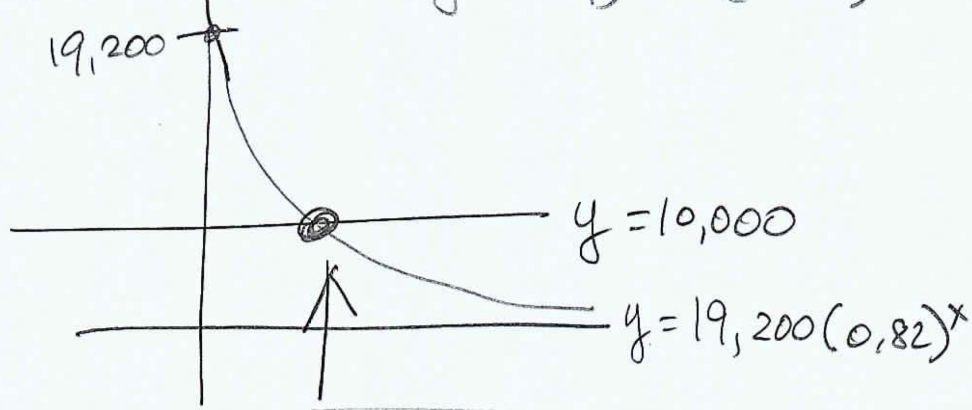
$t = ?$

$V(t) = 10{,}000$

$$V(t) = 19{,}200\,(0.82)^t$$

$$10{,}000 = 19{,}200\,(0.82)^t$$

$y_1 = 10{,}000$ $\qquad$ $y_2 = 19{,}200(0.82)^x$



$$\boxed{x \approx 3.3 \text{ years}}$$