College algebra

Class notes

Matrices and Systems of Equations (section 8.2)

Recall, we solved the system below in a previous section. Here, we learn another method.

$$x + 4y = 14$$
$$5x + 3y = 2$$

Back In

$$-5x - 20y = -70$$
$$5x + 3y = 2$$

Before, we solved this system by the elimination method. The matrix method has similarities and can be thought of as a way to shortcut the writing. This method is extremely useful when the system has lots of equations and variables and so the previous methods are cumbersome. Here is how we will write the matrix that represents this system.

$$\begin{bmatrix} 1 & 4 & | & 14 \\ 5 & 3 & | & 2 \end{bmatrix}$$

The first column are the x coefficients. The second column are the y coefficients. The constants make the third column. The vertical line represents the equal signs.

Terminology:

A rectangular array of numbers is a **matrix**. The plural is **matrices**. The numbers in a matrix are **entries** or **elements**.

The matrix $\begin{bmatrix} 1 & 4 \\ 5 & 3 \end{bmatrix}$ (which are the coefficients of the x and y terms) is called a **coefficient**

matrix. The matrix $\begin{bmatrix} 1 & 4 & | & 14 \\ 5 & 3 & | & 2 \end{bmatrix}$ (that represents the whole system of equations) is called an

augmented matrix.

Rows are horizontal and **columns** are vertical. A matrix that has m rows and n columns is said to be of **order** $m \times n$ (read "m by n"). The entry a_{ij} would be the entry in the ith row and j^{th} column.

When a matrix has the same number of rows as columns, it is said to be **square**.

The **main diagonal** starts at the upper left entry and goes down diagonally to the right.

We will start with some basic matrix manipulation and then move on to solving systems of equations using matrices.
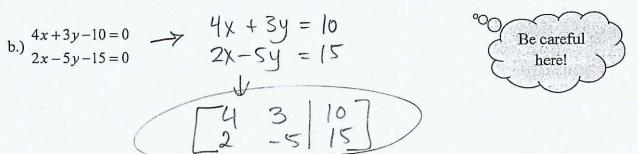
1

expl 1: Determine the order of the matrix.

a.) $\begin{bmatrix} 3 & 45 & -5 & 0 \end{bmatrix}$ order : 1×4

b.) $\begin{bmatrix} 2 \\ 7 \\ -2 \\ 6 \end{bmatrix}$ order : 4×1

> order $m \times n$:
> count rows, then columns

c.) $\begin{bmatrix} 6 & 5 & 4 & 9 \\ -3 & 0 & 2 & 5 \\ 4 & 8 & -3 & 4 \end{bmatrix}$ order : 3×4

expl 2: Write the augmented matrix for the system of equations.

a.)
$$5x + 6y - 3z = 12$$
$$2x \quad\quad + z = 5$$
$$\quad -2y + 4z = -16$$

$$\Rightarrow \left[\begin{array}{ccc|c} 5 & 6 & -3 & 12 \\ 2 & 0 & 1 & 5 \\ 0 & -2 & 4 & -16 \end{array}\right]$$

b.)
$$4x + 3y - 10 = 0$$
$$2x - 5y - 15 = 0$$

$$\rightarrow \quad 4x + 3y = 10$$
$$2x - 5y = 15$$

> Be careful here!

$$\downarrow$$
$$\left[\begin{array}{cc|c} 4 & 3 & 10 \\ 2 & -5 & 15 \end{array}\right]$$

expl 3: This augmented matrix is in row-echelon form. Write the system of equations that corresponds to it.

$$\left[\begin{array}{cc|c} 1 & 3 & -8 \\ 0 & 1 & 4 \end{array}\right] \rightarrow$$

$$1x + 3y = -8$$
$$1y = 4$$

> Go further and find x and y.

To find x, plug 4 in for y into 1st equ:

$$1x + 3(4) = -8$$
$$x + 12 = -8$$
$$x = -20$$

$$\rightarrow \quad \boxed{\text{Soln: } (-20, 4)}$$

2

Solving These Systems:

To solve a system of equations, we will write and then solve the augmented matrix by either the **Gaussian elimination method** or the **Gauss-Jordan elimination method**. The difference is simply where you choose to stop and revert back to the equations to decipher the solution.

Think back to how you solved this system (from the top of page 1) with the elimination method. What would have been "legal" to do to the equations? We will perform matrix operations in much the same manner. The operations that we are allowed to do are called **row operations** and are described below. The point is to transform the matrix into **row-echelon form** or, more preferably, **reduced row-echelon form**. These are described below.

Row-Equivalent Operations (things you are allowed to do to the matrix):

1. Interchange any two rows.
2. Multiply each entry in a row by the same non-zero constant.
3. Add a non-zero multiple of a row to another row, replacing that row.

Row-Echelon Form (when you stop and read the solution to the system):

1. If a row does *not* consist entirely of 0's, then the first non-zero entry in the row is a 1 (called a **leading 1**).
2. For any two successive non-zero rows, the leading 1 in the lower row is farther to the right than the leading 1 in the higher row.
3. All the rows consisting entirely of 0's are at the bottom of the matrix.

If the following condition is also met, the matrix is said to be in **reduced row-echelon form**.
4. Each column that contains a leading 1 has 0's everywhere else.

An Example in Full:

We will solve the system $\begin{array}{l} x+4y=14 \\ 5x+3y=2 \end{array}$ by using the augmented matrix $\begin{bmatrix} 1 & 4 & | & 14 \\ 5 & 3 & | & 2 \end{bmatrix}$. Try to follow the completed example on the next page.

Remember that the first column are the coefficients of the x terms and the second column are the coefficients of the y terms. Look carefully at the way we denote row operations; this varies from book to book. We will work to get 1's on the main diagonal and 0's elsewhere.

$$\begin{bmatrix} 1 & 4 & | & 14 \\ 5 & 3 & | & 2 \end{bmatrix} \xrightarrow{R_2 = -5r_1 + r_2} \begin{bmatrix} 1 & 4 & | & 14 \\ 0 & -17 & | & -68 \end{bmatrix} \xrightarrow{R_2 = (-1/17)r_2} \begin{bmatrix} 1 & 4 & | & 14 \\ 0 & 1 & | & 4 \end{bmatrix}$$

1. Get 0 below the 1 in the first column.

2. Get all 1's on the main diagonal.

3. Get 0's above and below the 1's.

We want all 1's along the main diagonal and 0's elsewhere.

$$R_1 = -4r_2 + r_1$$

$$\begin{bmatrix} 1 & 0 & | & -2 \\ 0 & 1 & | & 4 \end{bmatrix}$$

4. Read the solution.
$1x + 0y = -2$
$0x + 1y = 4$

What are x and y?

Use R_1 to mean "new row 1" and r_1 to mean "old row 1".

Gaussian elimination versus Gauss-Jordan elimination:
If we stop with the matrix in row-echelon form, the third matrix in the sequence above, we have used Gaussian elimination. If we continue, as we did, to write the matrix in *reduced* row-echelon form, we are using Gauss-Jordan elimination. I prefer to go all the way but you do *not* have to.

expl 4: Solve the system using an augmented matrix. Do it by hand and check your answer. We will learn later how to do this on the calculator.
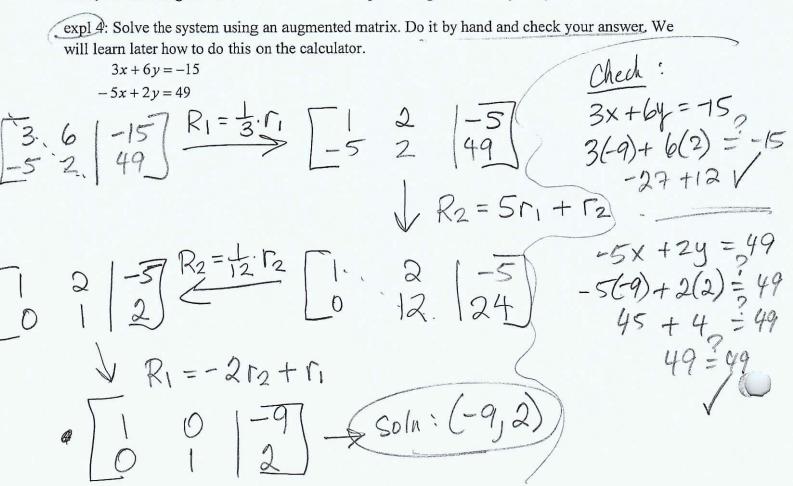
$3x + 6y = -15$
$-5x + 2y = 49$

$$\begin{bmatrix} 3 & 6 & | & -15 \\ -5 & 2 & | & 49 \end{bmatrix} \xrightarrow{R_1 = \frac{1}{3} \cdot r_1} \begin{bmatrix} 1 & 2 & | & -5 \\ -5 & 2 & | & 49 \end{bmatrix}$$

$$\downarrow R_2 = 5r_1 + r_2$$

$$\begin{bmatrix} 1 & 2 & | & -5 \\ 0 & 12 & | & 24 \end{bmatrix} \xrightarrow{R_2 = \frac{1}{12} \cdot r_2} \begin{bmatrix} 1 & 2 & | & -5 \\ 0 & 1 & | & 2 \end{bmatrix}$$

$$\downarrow R_1 = -2r_2 + r_1$$

$$\begin{bmatrix} 1 & 0 & | & -9 \\ 0 & 1 & | & 2 \end{bmatrix} \rightarrow \text{Soln} : (-9, 2)$$

Check :
$3x + 6y = -15$
$3(-9) + 6(2) \overset{?}{=} -15$
$-27 + 12 \checkmark$

$-5x + 2y = 49$
$-5(-9) + 2(2) \overset{?}{=} 49$
$45 + 4 \overset{?}{=} 49$
$49 \overset{?}{=} 49$ ✓

Using the Calculator:
We need to enter the augmented matrix into the calculator and then tell it to perform RREF (reduced row-echelon form) on the matrix. Follow the steps below to do the next example.

1. Enter the **MATRIX** menu. It is the **2nd** function of the x^{-1} button. Arrow over to **EDIT**.
2. Select **[A]** and press **ENTER**. You will then enter the order of the matrix, rows x columns.
3. Fill in the matrix with its entries.
4. Quit out to the home screen and re-enter the **MATRIX** menu. This time arrow over to **MATH**. Select **B: rref(**. This is at the bottom of the list under the numbered items.
5. Re-enter the **MATRIX** menu and select **[A]** from the **NAMES** list.
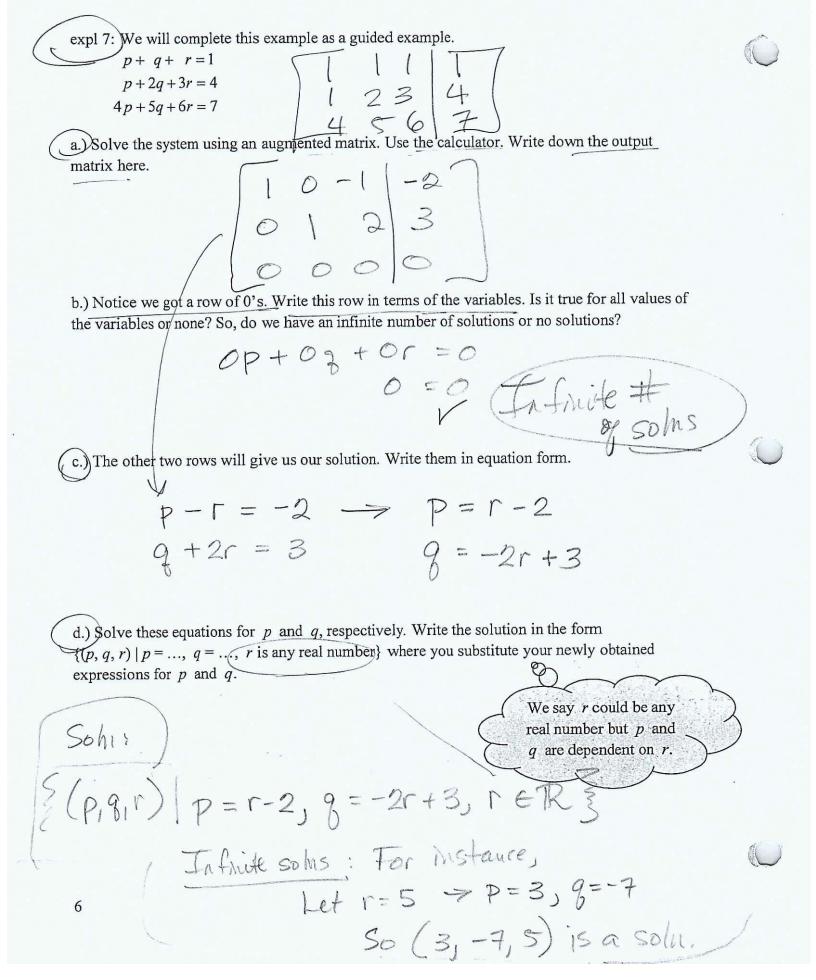6. Your home screen should now read **rref([A]**. Press **ENTER** and it will output the solved matrix.

expl 5: Solve the system using an augmented matrix. Use the calculator. Write the answer in ordered triplet form and check it.

$$\begin{cases} 3x + 2y + 2z = 3 \\ x + 2y - z = 5 \\ 2x - 4y + z = 0 \end{cases} \rightarrow \begin{bmatrix} 3 & 2 & 2 & | & 3 \\ 1 & 2 & -1 & | & 5 \\ 2 & -4 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1/2 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

$$x = 2, \quad y = 1/2, \quad z = -2 \quad \text{OR} \quad \left(2, \tfrac{1}{2}, -2\right)$$

Check: $3x + 2y + 2z = 3$
$3(2) + 2(\tfrac{1}{2}) + 2(-2) = 3$ ✓

$x + 2y - z = 5$?
$2 + 2(\tfrac{1}{2}) + 2 = 5$ ✓

$2x - 4y + z = 0$?
$2(2) - 4(\tfrac{1}{2}) - 2 = 0$
$4 - 2 - 2 = 0$ ✓

Matrices with an Infinite Number of Solutions or No Solutions:

expl 6: Solve the system using an augmented matrix. Use the calculator.

$$\begin{cases} -2x + 6y = 4 \\ 3x - 9y = 5 \end{cases}$$

$$\begin{bmatrix} -2 & 6 & | & 4 \\ 3 & -9 & | & 5 \end{bmatrix}$$

↓ calculator

$$\begin{bmatrix} 1 & -3 & | & 0 \\ 0 & 0 & | & 1 \end{bmatrix}$$

↳ $0x + 0y = 1$

Convert the bottom line of your output to variable form. For what values of x and y would this be true? So, what is the solution to this system?

No Soln

expl 7: We will complete this example as a guided example.

$$p + q + r = 1$$
$$p + 2q + 3r = 4$$
$$4p + 5q + 6r = 7$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \end{bmatrix}$$

a.) Solve the system using an augmented matrix. Use the calculator. Write down the output matrix here.

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b.) Notice we got a row of 0's. Write this row in terms of the variables. Is it true for all values of the variables or none? So, do we have an infinite number of solutions or no solutions?

$$0p + 0q + 0r = 0$$
$$0 = 0 \checkmark \quad \text{Infinite \# of solns}$$

c.) The other two rows will give us our solution. Write them in equation form.

$$p - r = -2 \rightarrow p = r - 2$$
$$q + 2r = 3 \qquad q = -2r + 3$$

d.) Solve these equations for p and q, respectively. Write the solution in the form $\{(p, q, r) \mid p = ..., q = ..., r$ is any real number$\}$ where you substitute your newly obtained expressions for p and q.

We say r could be any real number but p and q are dependent on r.

Soln:

$$\{(p, q, r) \mid p = r - 2, q = -2r + 3, r \in \mathbb{R}\}$$

Infinite solns: For instance,
Let $r = 5 \rightarrow p = 3, q = -7$
So $(3, -7, 5)$ is a soln.

6

expl 8: Let's practice again with row operations by hand. Perform the following row operations on the given augmented matrix.

$$\begin{bmatrix} 1 & -3 & -4 & | & -6 \\ 6 & -5 & 6 & | & -6 \\ -1 & 1 & 4 & | & 6 \end{bmatrix} \quad R_2 = -6r_1 + r_2 \text{ and } R_3 = r_1 + r_3$$

$$\begin{bmatrix} 1 & -3 & -4 & | & -6 \\ 0 & 13 & 30 & | & 30 \\ 0 & -2 & 0 & | & 0 \end{bmatrix}$$

Worksheet: Solving Systems with Matrices:
We will practice row operations by hand and solving an augmented matrix with the calculator.

Occasionally, the book will give us systems that are *not* written with the variable terms conveniently lined up. Be careful when converting these to matrices.

expl 9: Solve the system using an augmented matrix. Use the calculator.

$$\begin{aligned} x - y &= 6 \\ 2x - 3z &= 16 \\ 2y + z &= 4 \end{aligned} \longrightarrow \begin{bmatrix} 1 & -1 & 0 & | & 6 \\ 2 & 0 & -3 & | & 16 \\ 0 & 2 & 1 & | & 4 \end{bmatrix}$$

$x \quad y \quad z$ calculator

$$\begin{bmatrix} 1 & 0 & 0 & | & 8 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \longrightarrow \underline{\text{Soln}} \quad (8, 2, 0)$$

7

$0x + 0y + 1z = 0 \longrightarrow (z = 0)$