

We will see basic operations on matrices as well as learn to use inverses to solve systems of equations.

College algebra

Class notes

Matrices: Matrix Algebra Including Inverses (section 8.4)

Recall, we will define a **matrix** to be a rectangular array of numbers. To make the operations easier, we will use the following as a generic matrix of order  $m \times n$  ( $m$  rows and  $n$  columns).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & a_{ij} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{m1} & \dots & \dots & \dots & \dots & \dots & a_{mn} \end{bmatrix}$$

m rows

n columns

Every entry has a subscript describing the row and column it is in. The entry  $a_{ij}$  would be the entry in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column.

We say the entry has two indices, the row index  $i$  and the column index  $j$ .

Recall, when a matrix has the same number of rows as columns, it is said to be square. We might denote this matrix with order  $m \times m$ .

We need definitions for equal matrices as well as formulas for find the sum, product, and difference of matrices. We will also see something called the scalar multiple of a matrix. We will see that matrices share some properties of real numbers (but *not* all). Lastly, we will explore the inverse of a matrix and how it can be used to solve a system of equations using Gauss-Jordan elimination (mostly) as before.

These first ones will likely not surprise you.

**Definition: Equal Matrices:**

Two matrices are equal (we will write  $A = B$ ) when they have the same number of rows and the same number of columns and their corresponding entries are equal. (We may say this last bit as  $a_{ij}$  in  $A$  is equal to  $b_{ij}$  in  $B$ .)

**Definition: Sum and Difference of Matrices:**


You can only add or subtract two matrices if they have the same order. If they do, add or subtract the corresponding entries to form a new matrix and, poof, you have their sum or difference. We will denote these as  $A + B$  and  $A - B$ .



**Definition: Scalar Multiple of a Matrix:**

To multiply a matrix by a scalar, denoted by  $kA$ , we will multiply each entry of matrix  $A$  by the number  $k$ .

A scalar is a constant with magnitude but not direction. For instance, speed is scalar; you can go 60 mph. Velocity is a vector quantity; it has implied direction too; you go 60 mph down.



expl 1: For the following matrices, do the operations indicated. If it is *not* possible, label it as undefined.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}^{2 \times 2} \quad B = \begin{bmatrix} 0 & 7 \\ 4 & -3 \end{bmatrix}^{2 \times 2} \quad C = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix}^{2 \times 3}$$

a.)  $A + B = \begin{bmatrix} 3 & 11 \\ 2 & 2 \end{bmatrix}$

b.)  $\underline{3A} - B = \begin{bmatrix} 9 & 12 \\ -6 & 15 \end{bmatrix} - \begin{bmatrix} 0 & 7 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ -10 & 18 \end{bmatrix}$

c.)  $A - C$    
  $\downarrow$  *d.a.e.* or undefined

We will see that commutativity of multiplication,  $AB = BA$  does *not* hold up!

**Properties of Matrices (Similar to Real Numbers):**

Commutative Property of Addition:  $A + B = B + A$

Associative Property of Addition:  $(A + B) + C = A + (B + C)$

Associative Property of Multiplication:  $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

Distribution Property:  $A(B + C) = A \cdot B + A \cdot C$

Additive Identity:  $A + 0 = 0 + A = A$

But wait! What is 0 in the world of matrices?

matrix with all 0's

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

### Multiplication of Matrices:

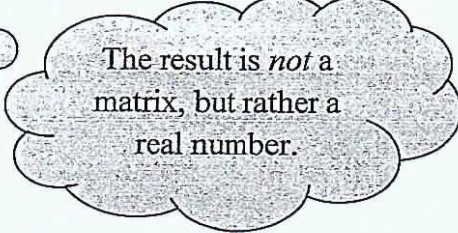
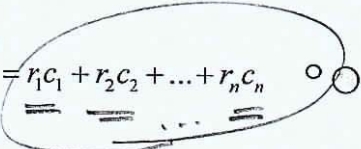
Multiplying two matrices is not as cut-and-dry as adding or scalar multiplication. In fact, you cannot multiply just any old matrices. First, we define the multiplication of a matrix with just one row by another matrix with just one column. The matrices need to have the same number of entries.

### Product of a Row Vector and a Column Vector:

A row vector  $R$  is a  $1 \times n$  matrix. A column vector  $C$  is an  $n \times 1$  matrix.

We will write them as  $R = [r_1 \ r_2 \ \dots \ r_n]$  and  $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ .

The product  $RC$  of “ $R$  times  $C$ ” is defined as the following number.

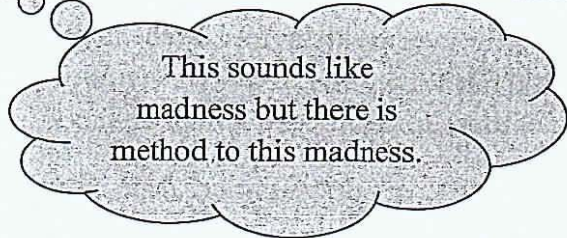
$$RC = [r_1 \ r_2 \ \dots \ r_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \underline{r_1 c_1 + r_2 c_2 + \dots + r_n c_n}$$


We define the product of two matrices in terms of the products of these row and column vectors.

### Product of Two Matrices:

rows x columns

Let  $A$  be an  $m \times r$  matrix and let  $B$  be an  $r \times n$  matrix. The product  $AB$  is defined as the  $m \times n$  matrix whose entry in row  $i$  and column  $j$  is the product (defined above) of the  $i^{\text{th}}$  row of  $A$  and the  $j^{\text{th}}$  column of  $B$ .



It is important to note that this would not work unless the number of columns in  $A$  was the same as the number of rows in  $B$ . That is why we see those two values are both  $r$  in the definition above.

It is a good idea to write the order of the matrices you are about to multiply together and see if the inside numbers match. We will practice this in the next example.



8.4

rows x columns

expl 2: For the following matrices, do the operations indicated. If it is *not* possible, label it as undefined.

$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$   $2 \times 2$     
  $B = \begin{bmatrix} 0 & 7 \\ 4 & -3 \end{bmatrix}$   $2 \times 2$     
  $C = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix}$   $2 \times 3$

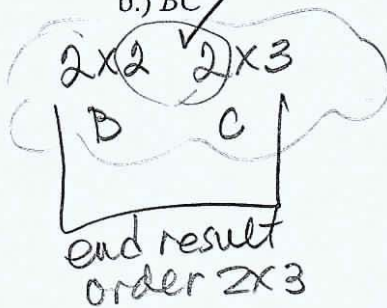
a.)  $AB$

$\begin{matrix} 2 \times 2 & 2 \times 2 \\ A & B \end{matrix}$ 
 $AB = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 0 & 7 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 16 & 9 \\ 20 & -29 \end{bmatrix}$

end result (product) has order  $2 \times 2$

$3 \cdot 0 + 4 \cdot 4 = 16$   
 $-2 \cdot 0 + 5 \cdot 4 = 20$   
 $3 \cdot 7 + 4(-3) = 9$   
 $-2 \cdot 7 + 5(-3) = -14 - 15 = -29$

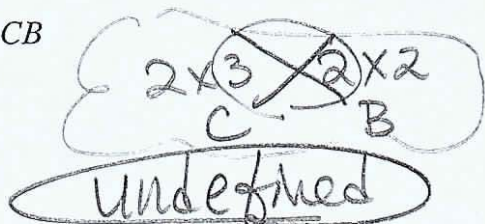
b.)  $BC$



$BC = \begin{bmatrix} 0 & 7 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 35 \\ 13 & -8 & -3 \end{bmatrix}$

$0 \cdot 4 + 7 \cdot 1 = 7$       $4(4) + -3(1) = 13$   
 $0(-2) + 7(0) = 0$       $4(-2) + -3(0) = -8$   
 $0(3) + 7(5) = 35$       $4(3) + -3(5) = -3$

c.)  $CB$



$5 \cdot 1 = 5$  "1"

**Multiplicative Identity Matrix:**

Remember how every real number can be multiplied by 1 and you just get the number you started with? That number 1 is called the multiplicative identity of the real numbers. Do we have one of those for matrices? You betcha!

Since being able to multiply matrices depends on their orders, this identity needs to have a somewhat flexible order. It so happens that if we use a square matrix with 1's along the main diagonal and 0's elsewhere, and an appropriate order, that will serve as our identity matrix.

We will use subscripts to denote the number of rows (and columns, 'cause they will be the same)

for the identity matrix. So, we have  $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ,  $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ , etc.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

expl 3: Multiply  $A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$  by the identity for  $2 \times 2$  matrices to show you just get  $A$ . In other words, find  $I_2 A$  and  $A I_2$ .

$$I_2 \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = A$$

$$\begin{aligned} 1 \cdot 3 + 0(-2) &= 3 \\ 0 \cdot 3 + 1(-2) &= -2 \\ 1 \cdot 4 + 0(5) &= 4 \\ 0 \cdot 4 + 1(5) &= 5 \end{aligned}$$

and  $A \cdot I_2 = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} = A$

We saw this in the last example.

**Properties for Identity Matrices:**

- 1) If  $A$  is an  $m \times n$  matrix, then we know  $I_m A = A$  and  $A I_n = A$ .
- 2) Let  $A$  be a square  $n \times n$  matrix. If there exists an  $n \times n$  matrix  $A^{-1}$  (read " $A$  inverse") for which  $AA^{-1} = A^{-1}A = I_n$ , then  $A^{-1}$  is called the **inverse of the matrix  $A$** .

Remember in real numbers, how  $\frac{1}{4} \cdot 4 = 1$ ?  
This is the matrix equivalent.

**Finding the Inverse of a Matrix (if it Exists):**

Not every matrix has an inverse. First, the matrix must be square, but not every square matrix has an inverse. When a square matrix does have an inverse, we say it is **nonsingular**. If a matrix does not have an inverse, it is said to be **singular**.

We find the inverse (if it exists) by forming an augmented matrix with the identity of the same order (original matrix on left and identity on right). Then, we convert it to reduced row echelon form (perform RREF on the calculator). The identity will end up on the left side and the inverse will be on the right.

expl 4: Use the calculator to find the inverse of the nonsingular matrix  $\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$ . We'll start with

the augmented matrix  $\begin{bmatrix} 6 & 5 & | & 1 & 0 \\ 2 & 2 & | & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & | & 1 & -5/2 \\ 0 & 1 & | & -1 & 3 \end{bmatrix}$

So, the inverse of  $\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$  is  $\begin{bmatrix} 1 & -5/2 \\ -1 & 3 \end{bmatrix}$ .



### Solving a System of Equations Using Inverses:

If we set up a system of equations in terms of matrices, more explicitly than we have done in the past, we can see how to use the inverse of a matrix to solve the system.

Consider the system of equations  $\begin{cases} 5x + 6y - 3z = 12 \\ 2x + z = 5 \\ -2y + 4z = -16 \end{cases}$ . We can see this as  $AX = B$  if we define

$A = \begin{bmatrix} 5 & 6 & -3 \\ 2 & 0 & 1 \\ 0 & -2 & 4 \end{bmatrix}$ ,  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ , and  $B = \begin{bmatrix} 12 \\ 5 \\ -16 \end{bmatrix}$ . Then, we could solve  $AX = B$  in the following manner.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ I_3X &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

So, if we can find the inverse of  $A$  and multiply that by  $B$ , we should get the system's solution.

Solve  
 $4x = 20$   
 $\frac{1}{4} \cdot 4x = \frac{1}{4} \cdot 20$   
 $x = 5$   
 $X = 5$

order matters

expl 5: Find the inverse of the matrix  $A$  above and multiply it (in the correct order) to  $B$  to find the solution of the system.

$X = A^{-1}B$  is solu to system

To find  $A^{-1}$ :

$$\left[ \begin{array}{ccc|ccc} 5 & 6 & -3 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{RREF}} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -1/13 & 9/13 & -3/13 \\ 0 & 1 & 0 & 4/13 & -10/13 & 11/26 \\ 0 & 0 & 1 & 2/13 & -5/13 & 6/13 \end{array} \right]$$

$\downarrow A^{-1}$

To find system's solu:

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 81/13 \\ -90/13 \\ -97/13 \end{bmatrix}$$

So, solu to system is  $\left( \frac{81}{13}, \frac{-90}{13}, \frac{-97}{13} \right)$ .

~~$B \cdot A^{-1}$~~

### When a Matrix Has No Inverse:

There are square matrices that have no inverse. We will know if we try to do the previous procedure and the identity does not appear on the left hand side. We will call this matrix singular.

expl 6: Show that the matrix  $\begin{bmatrix} 15 & 3 \\ 10 & 2 \end{bmatrix}$  has no inverse. In other words, perform RREF on the

augmented matrix  $\begin{bmatrix} 15 & 3 & | & 1 & 0 \\ 10 & 2 & | & 0 & 1 \end{bmatrix}$ .

↓ RREF

$$\left[ \begin{array}{cc|cc} 1 & 1/5 & 0 & 1/10 \\ 0 & 0 & 1 & -3/2 \end{array} \right]$$

↑

The left side is not the identity  $I_2$ .  
So the right side is not  
the orig. matrix' inverse.

The process failed  $\rightarrow$  So  
the matrix  $\begin{bmatrix} 15 & 3 \\ 10 & 2 \end{bmatrix}$  has  
no inverse.