We will use our equation solving skills to attack real-life problems.

College algebra
Class notes
Exponential and Logistic Growth and Decay Models; Newton's Law (section 6.8)

We will see various applications in this section. Keep in mind what the variables are said to represent and you'll be fine. We will, of course, need to use our equation solving skills we have developed through the semester.

Application: Population growth and decay:
These problems may refer to a population of people or animals but also money or anything else that grows or decays exponentially.

If a population (be it a population of a city, or number of deer in a certain area, or money in a bank account) **grows at a fixed rate each year** (or day, or second, etc.), then it is said to be **growing exponentially.** Likewise, **if a population decreases by a fixed rate each year** (or day, or second, etc.), it is said to be **decreasing or decaying exponentially**.

Formula for Population Growth and Decay:

We have $A(t) = A_0 e^{k \cdot t}$ with the following definitions.

A_0 = initial amount or population (at time 0)

$A(t)$ = amount or population after t years, days, etc.

t = time (years, days, etc.)

k = growth / decay constant

k is specific to situation:
if $k < 0$, decay
if $k > 0$, growth

[You may see $A(t) = A_0 e^{-kt}$ given as the formula for decay (where k is considered to be positive). I think it's easier to use one formula, taking k itself to be positive for growth and negative for decay.]

The book also gives a related formula, $N(t) = N_0 e^{k \cdot t}$, where N is the **number of cells in a culture** after time t (in the early stages of growth). Here, we assume growth and so $k > 0$.

expl 1: A population of rabbits has an exponential growth rate of 10.5 % per day. This population is given by $P(t) = 100e^{0.105t}$.

We put 10.5% in for k in decimal form.

a.) Find and interpret $P(0)$.

b.) Graph the function.

c.) What will the population be after 4 days?

d.) When will the population double?

What is the population when it has doubled?

e.) When will the population reach 1,000 rabbits?

Application: Radioactivity and Half-life:

The **half-life** of a substance is the amount of time it takes for one-half of the substance to decay. The half-life of radium is 1690 years. Consider the table below.

t (years)	amount of radium (grams)
0	100
1690	50
3380	25
5070	12.5
6760	6.25

At the beginning of time, we have 100 grams.

Every 1690 years, the amount present decreases by half.

expl 2: Iodine-131 is a radioactive material that decays according to the function $P(t) = P_0 e^{-.087t}$ where P_0 is the initial amount present and $P(t)$ is the amount present at time t (in days). (The **decay rate** of iodine-131 is 8.7 % per day. That information is already installed in the given formula as k. Notice it is negative.) Assume a scientist has a sample of 100 grams of iodine-131.

a.) Graph the function.

b.) How much is left after 9 days?

c.) When will there be 70 grams left?

d.) Find the half-life of Iodine-131.

If we started with 100 grams, what value should we use for $P(t)$ to find the half-life?

Application: Newton's Law of Cooling:
This law states that the temperature of a heated object decreases exponentially over time toward the temperature of the surrounding medium. Think about a cup of coffee on the counter; its temperature will decrease until it is near the temperature of the room.

The temperature u of a heated object at a given time t can be modeled by the function $u(t) = T + (u_0 - T)e^{k \cdot t}, \quad k < 0$ where T is the constant temperature of the surrounding medium, u_0 is the initial temperature of the heated object, and k is a negative constant.

expl 3: A thermometer reading 72° Fahrenheit is placed in a refrigerator where the temperature is a constant 38° F. After 2 minutes, the thermometer reads 60° F.
a.) Put the initial information into the formula to make a function for the thermometer's temperature in terms of k and t. Then, use the information "After 2 minutes, the thermometer reads 60° F." to find k. Finally, write the thermometer's temperature as a function of t.

b.) Graph the relationship between time and the thermometer's temperature. How long will it take the thermometer to reach 39° F ?

d.) The graph of this function has a horizontal asymptote, just as $y = e^x$ does. What is that asymptote? In other words, as time goes on, what does the thermometer's temperature approach?

Application: Logistic Model:
Uninhibited growth (like cell division) is eventually limited by factors such as living space and food. The logistic model is a more realistic model for decay or growth to be used in real life.

In a logistic model, the population P after time t is given by the function $P(t)=\dfrac{c}{1+ae^{-b\cdot t}}$

where a, b, and c are constants with $a>0$ and $c>0$. The model is a growth model if $b>0$ and a decay model if $b<0$.

The value $|b|$ is the **growth rate** for $b>0$ and the **decay rate** for $b<0$.

For growth models, the number c is called the **carrying capacity**. You will see that as t approaches infinity, $P(t)$ approaches c.

Properties of the Logistic Model:
1. The domain is all real numbers. The range is the interval $(0, c)$.
2. There are no x-intercepts. The y-intercept is $P(0)$.
3. There are two horizontal asymptotes, $y=0$ and $y=c$.
4. If $b>0$, the graph increases. If $b<0$, the graph decreases.
5. There is an inflection point where $P(t)$ equals one-half of the carrying capacity. (This means it changes from concave up to concave down or vice versa.)
6. The graph is smooth and continuous, with no corners or gaps.

As you read these properties, graph the function $y=\dfrac{8}{1+2e^{-3x}}$ on the window [-5, 8] x [-2, 10]. Then delete the negative sign to make $b<0$.

expl 4: The logistic model $W(t) = \dfrac{14,656,248}{1 + 0.059e^{0.057t}}$ represents the number of farm workers in the United States t years after 1910.

a.) Find and interpret $W(0)$.

b.) Graph this function.

> Recall, that $y = c$ will be a horizontal asymptote.
> Here, $b = -0.057$.
> Do you see why?

c.) How many farm workers were in the US in 2010?

d.) When did the number of farm workers reach 10,000,000? Solve graphically.