We will see basic operations on matrices as well as learn to use inverses to solve systems of equations.

College algebra
Class notes
Matrices: Matrix Algebra Including Inverses (section 8.4)

Recall, we will define a **matrix** to be a rectangular array of numbers. To make the operations easier, we will use the following as a generic matrix of order m x n (m rows and n columns).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & a_{ij} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix}$$

Every entry has a subscript describing the row and column it is in. The entry a_{ij} would be the entry in the i^{th} row and j^{th} column.

We say the entry has two indices, the **row index** i and the **column index** j.

Recall, when a matrix has the same number of rows as columns, it is said to be **square**. We might denote this matrix with order m x m.

We need definitions for **equal** matrices as well as formulas for find the **sum**, **product**, and **difference** of matrices. We will also see something called the **scalar multiple** of a matrix. We will see that matrices share some **properties** of real numbers (but *not* all). Lastly, we will explore the **inverse** of a matrix and how it can be used to solve a system of equations using Gauss-Jordan elimination (mostly) as before.

These first ones will likely not surprise you.

Definition: Equal Matrices:
Two matrices are **equal** (we will write $A = B$) when they have the same number of rows and the same number of columns and their corresponding entries are equal. (We may say this last bit as a_{ij} in A is equal to b_{ij} in B.)

Definition: Sum and Difference of Matrices:
You can only add or subtract two matrices if they have the same order. If they do, add or subtract the corresponding entries to form a new matrix and, poof, you have their **sum** or **difference**. We will denote these as $A + B$ and $A - B$.

Definition: Scalar Multiple of a Matrix:
To multiply a matrix by a scalar, denoted by kA, we will multiply each entry of matrix A by the number k.

A **scalar** is a constant with magnitude but *not* direction. For instance, speed is scalar; you can go 60 mph. Velocity is a **vector** quantity; it has implied direction too; you go 60 mph *down*.

expl 1: For the following matrices, do the operations indicated. If it is *not* possible, label it as undefined.

$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 & 7 \\ 4 & -3 \end{bmatrix}$ $\quad C = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix}$

a.) $A + B$

b.) $3A - B$

c.) $A - C$

Properties of Matrices (Similar to Real Numbers):

We will see that commutativity of multiplication, $AB = BA$ does *not* hold up!

Commutative Property of Addition: $A + B = B + A$

Associative Property of Addition: $(A + B) + C = A + (B + C)$

Associative Property of Multiplication: $(A \cdot B)\, C = A\, (B \cdot C)$

Distribution Property: $A\,(B + C) = A \cdot B + A \cdot C$

Additive Identity: $A + 0 = 0 + A = A$

But wait! What is 0 in the world of matrices?

Multiplication of Matrices:
Multiplying two matrices is *not* as cut-and-dry as adding or scalar multiplication. In fact, you *cannot* multiply just any old matrices. First, we define the multiplication of a matrix with just one row by another matrix with just one column. The matrices need to have the same number of entries.

Product of a Row Vector and a Column Vector:
A **row vector** R is a $1 \text{ x } n$ matrix. A **column vector** C is an $n \text{ x } 1$ matrix.

We will write them as $R = \begin{bmatrix} r_1 & r_2 & \ldots & r_n \end{bmatrix}$ and $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$.

The **product** RC of "R times C" is defined as the following number.

$$RC = \begin{bmatrix} r_1 & r_2 & \ldots & r_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \ldots + r_nc_n$$

The result is *not* a matrix, but rather a real number.

We define the product of two matrices in terms of the products of these row and column vectors.

Product of Two Matrices:
Let A be an $m \text{ x } r$ matrix and let B be an $r \text{ x } n$ matrix. The product AB is defined as the $m \text{ x } n$ matrix whose entry in row i and column j is the product (defined above) of the i^{th} row of A and the j^{th} column of B.

This sounds like madness but there is method to this madness.

It is important to note that this would *not* work unless the number of columns in A was the same as the number of rows in B. That is why we see those two values are both r in the definition above.

It is a good idea to write the order of the matrices you are about to multiply together and see if the inside numbers match. We will practice this in the next example.

expl 2: For the following matrices, do the operations indicated. If it is *not* possible, label it as undefined.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 7 \\ 4 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 0 & 5 \end{bmatrix}$$

a.) AB

b.) BC

c.) CB

Multiplicative Identity Matrix:
Remember how every real number can be multiplied by 1 and you just get the number you started with? That number 1 is called the **multiplicative identity** of the real numbers. Do we have one of those for matrices? You betcha!

Since being able to multiply matrices depends on their orders, this identity needs to have a somewhat flexible order. It so happens that if we use a square matrix with 1's along the main diagonal and 0's elsewhere, and an appropriate order, that will serve as our identity matrix.

We will use subscripts to denote the number of rows (and columns, 'cause they will be the same) for the identity matrix. So, we have $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, etc.

expl 3: Multiply $A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$ by the identity for 2x2 matrices to show you just get A. In other words, find $I_2 A$ and AI_2.

We saw this in the last example.

Properties for Identity Matrices:

1. If A is an m x n matrix, then we know $I_m A = A$ and $AI_n = A$.

2. Let A be a square n x n matrix. If there exists an n x n matrix A^{-1} (read "*A* inverse") for which $AA^{-1} = A^{-1}A = I_n$, then A^{-1} is called the **inverse of the matrix *A***.

Remember in real numbers, how $\frac{1}{4} \cdot 4 = 1$?

This is the matrix equivalent.

Finding the Inverse of a Matrix (if it Exists):

Not every matrix has an inverse. First, the matrix must be *square*, but *not* every square matrix has an inverse. When a square matrix does have an inverse, we say it is **nonsingular**. If a matrix does *not* have an inverse, it is said to be **singular**.

We find the inverse (if it exists) by forming an augmented matrix with the identity of the same order (original matrix on left and identity on right). Then, we convert it to reduced row echelon form (perform RREF on the calculator). The identity will end up on the left side and the inverse will be on the right.

expl 4: Use the calculator to find the inverse of the nonsingular matrix $\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$. We'll start with the augmented matrix $\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]$.

Solving a System of Equations Using Inverses:
If we set up a system of equations in terms of matrices, more explicitly than we have done in the past, we can see how to use the inverse of a matrix to solve the system.

Consider the system of equations $\begin{aligned} 5x + 6y - 3z &= 12 \\ 2x \qquad + z &= 5 \\ -2y + 4z &= -16 \end{aligned}$. We can see this as $AX = B$ if we define

$A = \begin{bmatrix} 5 & 6 & -3 \\ 2 & 0 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 12 \\ 5 \\ -16 \end{bmatrix}$. Then, we could solve $AX = B$ in the following manner.

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ I_3X &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

So, if we can find the inverse of A and multiply that by B, we should get the system's solution.

expl 5: Find the inverse of the matrix A above and multiply it (in the correct order) to B to find the solution of the system. Use the "Convert to Fraction" function under the MATH menu for your final answers.

The x^{-1} button (whose second function is MATRIX) will get us the inverse.

When a Matrix Has No Inverse:
There are square matrices that have *no* inverse. We will know if we try to do the previous procedure and the identity does *not* appear on the left hand side. We will call this matrix **singular**.

expl 6: Show that the matrix $\begin{bmatrix} 15 & 3 \\ 10 & 2 \end{bmatrix}$ has no inverse. In other words, perform RREF on the augmented matrix $\left[\begin{array}{cc|cc} 15 & 3 & 1 & 0 \\ 10 & 2 & 0 & 1 \end{array}\right]$.