

We will review simplifying radicals, exponents and their rules, multiplying polynomials, factoring polynomials, greatest common denominators, and solving rational equations.

Pre-requisite Material for Chapter 3

Simplifying Radicals:

We will work with two important rules for radicals. We will write them for square roots but they work for any root (cube root, fourth root, etc.).

Product and Quotient Rules: For any real numbers a and b , we have $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

Likewise, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$. (This assumes that these radicals are real numbers. In the case of the second rule, the denominators are non-zero. Why?)

It's always a good idea to investigate rules instead of believing them blindly. Use the examples below to verify the rules for yourself.

expl 1: Putting 4 in for a and 25 in for b , the product rule becomes $\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25}$. Simplify each side separately to verify the rule.

This is just a matter of order of operations. Do you multiply before square rooting, or square root then multiply? The rule says it should not matter.

Multiply, and then take the square root.

Square root each, and then multiply them.

Were they equal? Does the rule hold true?

expl 2: Putting 64 in for a and 16 in for b , the quotient rule becomes $\sqrt{\frac{64}{16}} = \frac{\sqrt{64}}{\sqrt{16}}$. Simplify each side to verify the rule.

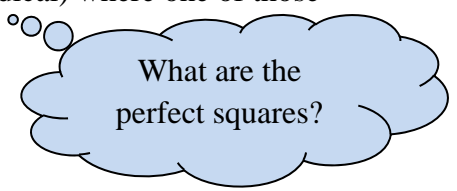
Divide, and then take the square root.

Square root each, and then divide them.

Were they equal? Does the rule hold true?

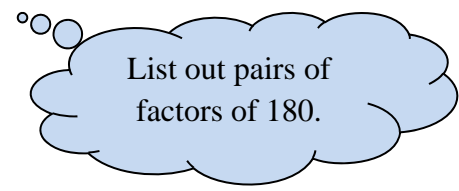
expl 3: We will practice these rules by simplifying the following radicals. The trick is to find a pair of numbers that multiply to the radicand (the number under the radical) where one of those numbers is a perfect square.

a.) Simplify $\sqrt{75}$.



What are the perfect squares?

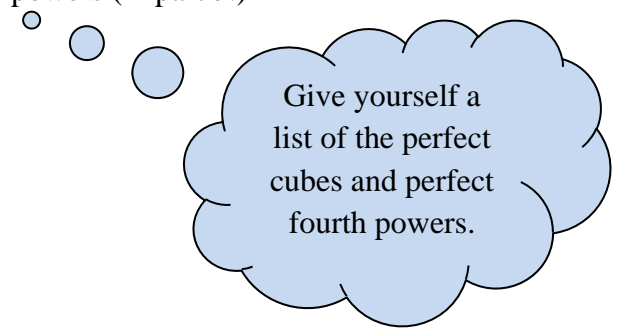
b.) Simplify $\sqrt{180}$.



List out pairs of factors of 180.

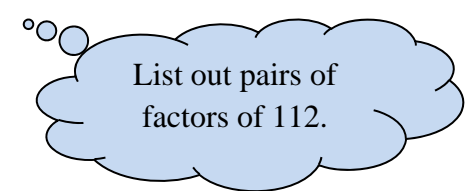
expl 4: Let's do a few with roots other than the square root. They work the same but, this time we will look for perfect cubes (in part *a*) and perfect fourth powers (in part *b*.)

a.) Simplify $\sqrt[3]{162}$.



Give yourself a list of the perfect cubes and perfect fourth powers.

b.) Simplify $\sqrt[4]{112}$.



List out pairs of factors of 112.

Exponents and Exponent Rules:

Recall an exponent's purpose. What does 5^3 really mean?

$$5^3 = 5 \cdot 5 \cdot 5$$

"5 multiplied by
itself 3 times"

Can be thought
of as "repeated
multiplication".

What about $(-2)^4$?

$$(-2)^4 = (-2)(-2)(-2)(-2)$$

"-2 multiplied by
itself 4 times"

What is the value of $(-2)^4$? How is that different than -2^4 ? The parentheses are very important.

We will be simplifying expressions like $\left(\frac{1}{3}\right)^4$ and $-4 \cdot 3^3$ and $(5y^4)(3y)$. Some can be done by simply thinking about what it means to raise a number by an exponent. Other times it will be easier to use the rules of exponents.

We will start with simplifying (or evaluating) expressions with no variables. Keep the order of operations in mind.

expl 1: Evaluate.

$$-4^3$$

PEMDAS
Parentheses
Exponents
Multiplication/**D**ivision
Addition/**S**ubtraction

expl 2: Evaluate.

$$(-4)^3$$

What difference do
the parentheses make?

Earlier we saw $(-2)^4$ and -2^4
were different. But $(-4)^3$ and
 -4^3 were the same. Why?

Worksheet: Discovering the Rules of Exponents

This worksheet will work on developing the rules from your knowledge of arithmetic and exponents. It will help internalize the rules. They will be easier to use and remember if you do not have to memorize them as meaningless drivel.

You will use the rules of exponents a lot. We may refer to them by name. Complete the rules from memory (after doing the worksheet) so that we can use them in the next examples. (The variables represent real numbers and denominators are not zero.)

Product rule: $a^m \cdot a^n =$

Quotient rule: $\frac{a^m}{a^n} =$

Power rule: $(a^m)^n =$

Power of a product rule: $(a \cdot b)^n =$

Power of a quotient rule: $\left(\frac{a}{c}\right)^n =$

Zero exponent rule: $a^0 =$ (Here a cannot be 0 because 0^0 is undefined.)

Make up quick
examples to help.

Think about $2^3 \cdot 2^4$.

$$2^3 \cdot 2^4 = (2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)$$

How many 2's is that? So
what is the product rule?

expl 3: Use the product rule to simplify each expression. Write the results using exponents.

$$x^2 \cdot x^5$$

expl 4: Use the product rule to simplify each expression. Write the results using exponents.

$$(-3)^3 \cdot (-3)^9$$

Do not worry about
the final value. Just
practice the rule.

expl 5: Use the product rule to simplify each expression. Write the results using exponents.

$$(5y^4)(3y)$$



Picture the constants
together and the
variables together.

$$5 \cdot 3 \cdot y^4 \cdot y = ?$$

expl 6: Use the power rule and the power of a product or quotient rule to simplify.

$$(x^9)^4$$



Rewrite the rule here
before you begin.

expl 7: Use the power rule and the power of a product or quotient rule to simplify.

$$(-7a^2b^5c)^2$$

expl 8: Use the power rule and the power of a product or quotient rule to simplify.

$$\left(\frac{xy}{7}\right)^2$$

Multiplying Polynomials:

Do you remember the distribution property? Write it down using variables or actual numbers.

The distribution property is my favorite because it is used so much. It gives us the ability to turn a product into a sum or turn a sum into a product. I wrote it down along with an example below. Did you remember it correctly?

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$3 \cdot (2 + 5) = 3 \cdot 2 + 3 \cdot 5$$

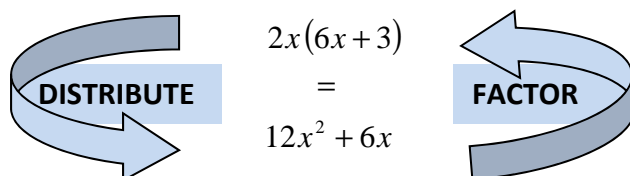
product

"3 times 7"

sum

"6 plus 15"

Notice how the two sides of the equation are in fact equal. Does your example work out too? We will use this property to multiply things like $2x(6x + 3)$. We would "distribute the $2x$ " to get our answer. In future sections, we will go the opposite way. That will be called "factoring". The relationship between the distribution property and factoring is illustrated below.



Try some examples to practice the distribution property.

expl 1: Multiply.

$$9t^6(-3t^5)$$

Why does the
distribution property
not apply here?

Remember
those exponent
rules!

expl 2: Multiply.

$$-2a^2(3a^2 - 2a + 3)$$

Simplify as
you go.

expl 3: Multiply.

$$\frac{1}{2}x^2(8x^2 - 6x + 1)$$

FOIL (First Outside Inside Last)

To understand that this is not some trick, **we will multiply $(x + 4)(x + 3)$ without relying on FOIL**. We will see how FOIL is simply a mnemonic device for this multiplication.

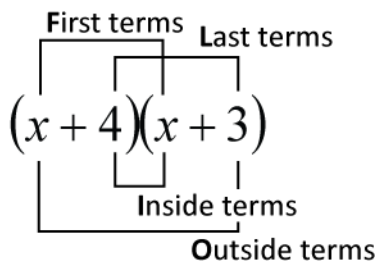
Multiply $(x + 4)(x + 3)$.

mnemonic device
used to multiply 2
binomials

Start by distributing
 $(x + 4)$ through the
whole $x + 3$.

Your first line should be
 $(x + 4) \cdot x + (x + 4) \cdot 3$.
How would you continue?

So $(x + 4)(x + 3) = \underline{x \cdot x} + \underline{4 \cdot x} + \underline{3 \cdot x} + \underline{3 \cdot 4}$. This happens to be the sum of the products of the **first** terms in each binomial, the **inside** terms, the **outside** terms, and the **last** terms in each binomial. See the picture below.



Simplify $(x + 4)(x + 3) = \underline{x \cdot x} + \underline{4 \cdot x} + \underline{3 \cdot x} + \underline{3 \cdot 4}$ to write the product in finished form.

In the future, use FOIL to multiply two binomials. Do it enough and it becomes second nature. Remember it does not matter which order you write your terms. Always simplify by combining any like terms to get the finished answer.

Worksheet: Understanding FOIL

This worksheet uses pictures to help you understand why FOIL works to multiply two binomials. It is a convincing argument that FOIL is needed for expressions like $(2x + 3)^2$.

expl 4: Use FOIL to multiply.

$$(a + 7)(a - 2)$$

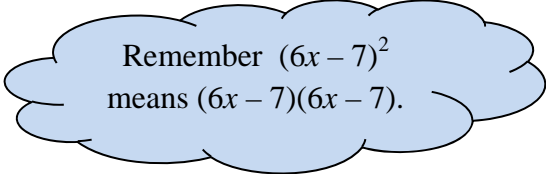
expl 5: Use FOIL to multiply.

$$(3x^2 + 1)(4x^2 + 7)$$

expl 6: Use FOIL to multiply.

$$(6x - 7)^2$$

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Remember $(6x - 7)^2$
means $(6x - 7)(6x - 7)$.

You may be asked to multiply a binomial and a trinomial or two trinomials. How would you work the following problems? We do not use FOIL per se, but we can think of it similarly.

expl 7: Multiply.

$$(x - 2)(x^2 - 3x + 7)$$

expl 8: Multiply vertically.

$$(x^2 + 5x - 7)(2x^2 - 7x - 9)$$

Everything in first trinomial gets multiplied by everything in second.

“Vertical” means to write your like terms below one another instead of all on one line. This makes it easier to combine like terms at the end.

expl 9: Multiply using FOIL.

$$(x - 5)(x + 10)$$

expl 10: Multiply using FOIL.

$$(2x - 9)(x - 11)$$

expl 11: Multiply.

$$(6a + 7)(6a + 5)$$

Factoring and Factoring Trinomials:

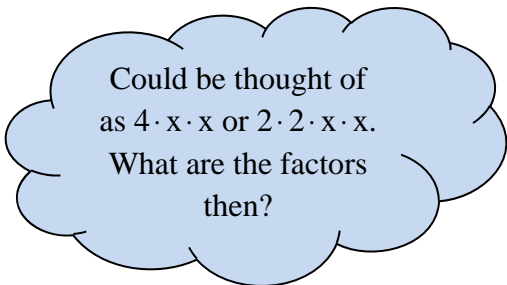
Factors versus terms:

terms: things we are adding (or subtracting)

expls: $\underline{x} + \underline{4}$ or $\underline{2x} + \underline{3}$ or $\underline{4x^2} + \underline{3x} - \underline{6}$

factors: things we are multiplying (or dividing)

expls: $\underline{5} \cdot \underline{x}$ or $\underline{3}(\underline{x+2})$ or $\underline{4} \cdot \underline{x^2}$ ○ ○ ○



Could be thought of
as $4 \cdot x \cdot x$ or $2 \cdot 2 \cdot x \cdot x$.
What are the factors
then?

Add your own examples to the lists above.

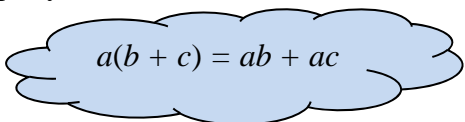
To factor an expression like $x^2 + 6x + 5$ means to **write it as a product** “something times something” **instead of a sum** “something plus something plus something”. Do you see how $x^2 + 6x + 5$ is a sum of three terms? What are those terms?

Recall that to factor 20, we write it as a product, $4 \cdot 5$ or $2 \cdot 2 \cdot 5$. To factor $x^2 + 6x + 5$, we also write it as a product. We will learn how to write $x^2 + 6x + 5$ as $(x+5)(x+1)$. We can think of this as the product of two factors. What are those factors?

Check that I am not lying to you. FOIL out $(x+5)(x+1)$ to make sure it really is $x^2 + 6x + 5$.

Another example: The expression $3x + 12$ can be thought of as a **sum** of the terms $3x$ and 12. But it is also equivalent to $3(x+4)$, which could be thought of as the **product** of two factors. What are they?

The fact that $3x + 12 = 3(x+4)$ is a lovely example of which property of real numbers?


$$a(b + c) = ab + ac$$

This will be used to factor expressions by drawing out the GCF (greatest common factor) from two (or more) terms like the 3 in this example.

To factor an expression:

1. Always start by factoring out the GCF from **all** terms if possible.
2. Various methods will be discussed later. If you are given a ...
 - 2-term expression like $x^2 - 16$, look into special formulas,
 - 3-term expression like $2x^2 - 5x - 3$, try the various methods given for trinomials,
 - 4-term expression like $2x^2 - 6x + 1x - 3$, try factoring by grouping.
3. After you factor, check the individual factors for additional factoring possibilities.
4. Always check your final answer by multiplying it out. Remember an expression and its factored form should be equivalent.
5. Some expressions are **prime**; they cannot be factored.

Greatest Common Factor (GCF):

The GCF of two numbers is the largest number that is a factor of both original numbers.

expl 1: Find the GCF of 10 and 45.

What are the factors
of 10 and 45?

10: 1, 2, 5, 10

45: 1, 3, 5, 9, 15, 45

Of the factors of 10 and 45, which is the largest common factor? That is the GCF.

expl 2: Find the GCF of $6x^2$ and $10x^3$.

Same thing here, but the x^2 and x^3
parts feel weird. What is the largest
factor they both contain?

x^2 is $x \cdot x$

x^3 is $x \cdot x \cdot x$

expl 3: Find the GCF of $6x^2y^5z^3$, $9xy^2z$, and $30x^2y^2z^3$.

Recall the Distribution Property:

It allows us to write a sum as a product, which is what factoring is all about. We use the distribution property to factor $3x + 12$ by pulling out the GCF of 3.

$$ab + ac = a(b + c)$$

$$3x + 12 = 3(x+4)$$

sum

product

expl 4: Factor the GCF out of the expression $18a^3 - 27a^5 + 9a^4$.

What is the GCF of the three terms? Pull it out in front using the Distribution Property.

Is your final answer written as a product? Check your answer by multiplying it back out.

expl 5: Factor $4(x + 2) + x(x + 2)$.

Here, see you have two terms, double-underlined below.

$$\underline{4(x + 2)} + \underline{x(x + 2)}$$

What is the common factor?
Factor it out like before.

expl 6: Factor by using the Distribution Property $14x^2y + 7xy - 21x$.

Solving Rational Equations:

Solving an equation means to rewrite it over and over until the variable is isolated, like $x = 4$. That will be the number (or numbers) that makes the original equation true. There are lots of different ways to do this depending on the type of equation. You know methods for linear equations (such as $3x + 4 = 10$) and quadratic equations (such as $x^2 - 7x - 60 = 0$). Here we learn how best to deal with rational equations like $\frac{x}{5} + \frac{1}{6} = \frac{8}{3}$ or $\frac{6x+13}{x-1} = \frac{4x+15}{x-1}$.

Main idea: The most efficient method is to multiply all fractions by their LCD. That eliminates the fractions as you will see. Make sure you always multiply **every term** by the LCD.

expl 1: Solve $\frac{x}{5} + \frac{1}{6} = \frac{8}{3}$.

What is the LCD
of 5, 6, and 3?

Multiply every
fraction by 30.

When you multiplied by 30, did the denominators in your fractions get eliminated?

$$30 \cdot \frac{x}{5} = ?$$

$$30 \cdot \frac{1}{6} = ?$$

$$30 \cdot \frac{8}{3} = ?$$

So our equation became $6x + 5 = 80$. Solve that as you would any linear equation.

Notice how each step gets
you closer to having x
alone. That is the whole
idea of solving an
equation.

Checking your answer:

A very important step in solving rational equations is checking your answer. It turns out that sometimes a solution actually turns out to be what is called an “extraneous solution”. Even though you get it by doing good algebra, it turns out that it **does not make the equation true** after all. Remember, the whole point is to find the value(s) of the variable that make the equation true.

Start with the original equation and see if 12.5 really is a solution. Does it make $\frac{x}{5} + \frac{1}{6} = \frac{8}{3}$ true?

I would personally do this on my calculator. Enter the following into your calculator.

Left side of equation

12.5 / 5 + 1 / 6

Right side of equation

8 / 3

Use division
key for
fractions.

Do they match?
Does 12.5 make the
equation true?

expl 2: Solve. Check your answer.

$$\frac{3x}{4} + \frac{5}{6} = \frac{x}{12} + \frac{7}{8}$$

Multiply all fractions
by the LCD.