We will learn the basics of polynomial functions and their graphs.

College algebra
Class notes
Polynomial Functions (section 5.1)

Definition: Polynomial function:
A polynomial function is a function that can be written in the form
$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \ldots + a_1 x + a_0$ where $a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are real numbers and the exponents are whole numbers.

No exponents can be fractions or negative numbers.

Do you see how these fit the general formula?

expls: $y = ax^2 + bx + c$

$f(x) = mx + b$

$y = \frac{1}{2}x^5 - 4x^4 + 2x^3 - 7$

$f(x) = \sqrt{5}x^3 + 4x^2 - x + 10$

$g(x) = \sqrt{2}x^2 - 5x^7 + 4x^6 - \sqrt{12}$

$y = (2x+3)(x-1)$

Standard form means the exponents are in decreasing order.

Some counterexamples follow. Can you tell why they are *not* polynomials? How do they *not* fit the definition?

counterexpls: $y = 4x^{-3} + 2x - 7$, $\quad f(x) = \frac{1}{2}\sqrt{x} + 4x^2$, $\quad y = \frac{14x^2 + x}{x^3 - 8}$

Terminology:

$a_n, a_{n-1}, a_{n-2}, \ldots, a_0$ are called **coefficients**

Consider $y = 4x^3 - 5x^2 + 8$. Identify these parts.

a_0 is called the **constant coefficient** (or **term**)

a_n is called the **leading coefficient**

$a_n x^n$ is called the **leading term**

n is called the **degree**

x is called the **variable**

Characteristics of Graphs of Polynomial Functions:

1.) The **domain** will always be “all real numbers”. (We have *no* square roots or division by 0 in polynomial functions. So nothing stands in the way of an x value “working” in the function.)

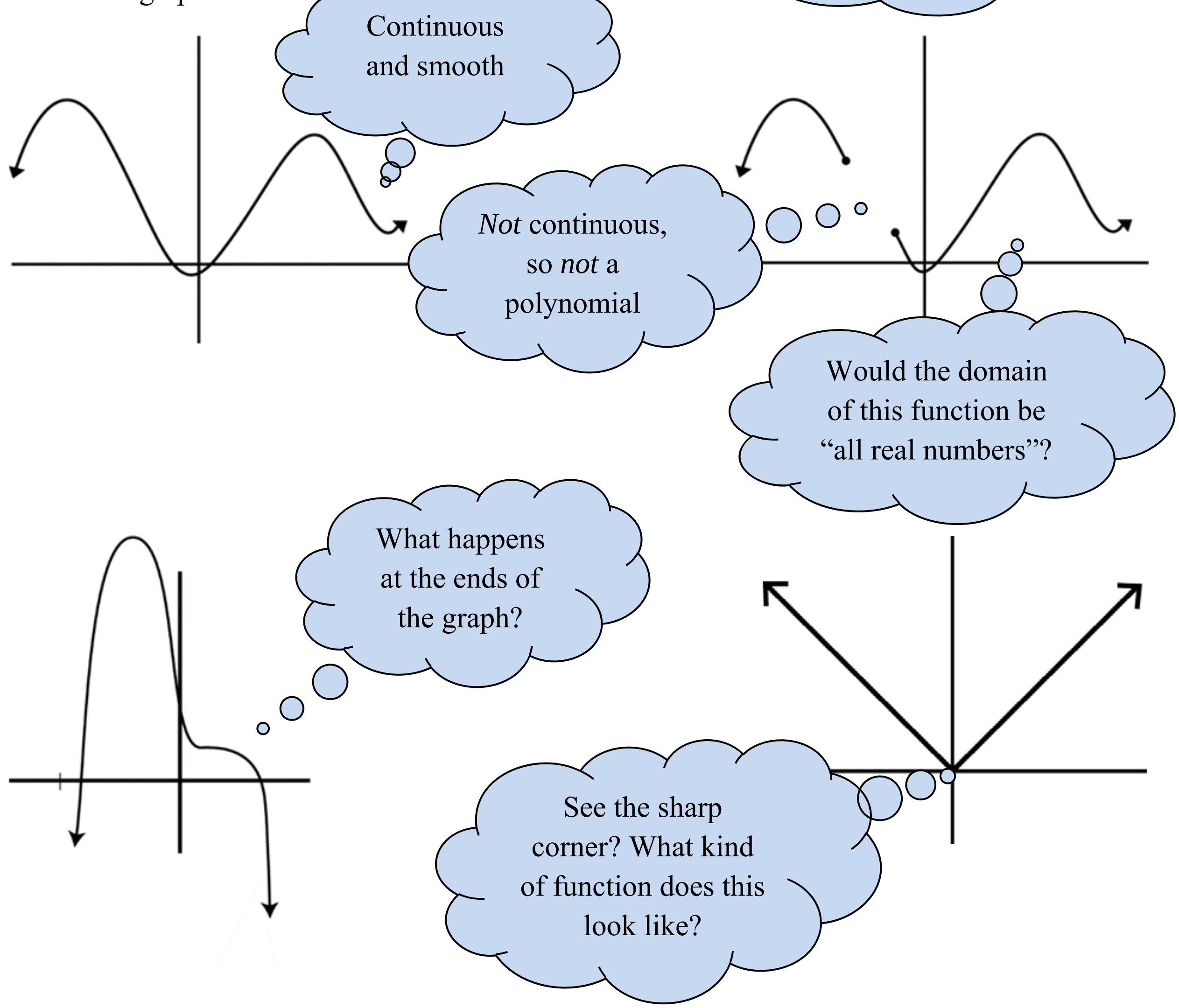
2.) The graph is **continuous**. This means you could trace the whole graph from the left end to the right end without lifting your pencil.

3.) The graph has **no sharp corners**. It is a smooth curve.

4.) The last major characteristic of a polynomial graph is its **end behavior**. End behavior answers the question, “what is happening to the y values at the (left and right) ends of the graph?” We’ll investigate end behavior next.

We could say “for large values of $|x|$”.

Look at the graphs below.



Worksheet: Polynomial functions: End behavior:
This worksheet explores the end behavior of the graphs of polynomial functions. We look at what is happening to the y values at the (left and right) ends of the graph. In other words, we are interested in what is happening to the y values as we get really large x values and as we get really small (negative) x values.

Fill in this table as a summary of what you learned from the worksheet. Recall how the leading term alone will determine a polynomial function's end behavior. This is sometimes referred to as the **Leading Term Test**.

	Leading coefficient is negative	**Leading coefficient is positive**
Degree is odd		
Degree is even		

expl 1: Determine the leading term, leading coefficient, constant term, and the degree of the polynomial.

a.) $r(x) = -5x^4 + 2x^3 - 7$

Leading term: term with highest exponent
Constant term: no x

b.) $f(x) = 3 - 5x^2 + 9x^3 - 6x^7 + 5x$

expl 2: Find the end behavior of each function. Write the end behavior using the notation shown in the worksheet "Polynomial functions: End behavior".

a.) $r(x) = -5x^4 + 2x^3 - 7$

b.) $f(x) = 3 - 5x^2 + 9x^3 - 6x^7 + 5x$

Worksheet: Polynomial functions: End behavior 2:
This worksheet will help you practice use the procedure and notation described here.

Definition: Power function: A **power function** is a polynomial function with only one term (called a monomial). In general, we write this as $y = ax^n$ where a is a real number *not* equal to 0 and n is an non-negative integer. Notice that n would be the **degree** of this function.

Connection Between Polynomial End Behavior and Power Functions:
Since the end behavior of a polynomial function is solely determined by the leading term, we can talk about the end behavior of a polynomial function f by referring to the power function that "resembles the graph of f for large values of $|x|$."

For instance, consider the function from the last page, $r(x) = -5x^4 + 2x^3 - 7$. Which "power function" would the graph of $r(x)$ resemble for large values of $|x|$?

Again, by "large values of $|x|$", we mean the ends of the graph.

If a function is given in factored form, multiply it all out to see it in standard form.

Zeros of Polynomial Functions:

Recall: Definitions: Zero and x-intercept:
An ***x*-intercept** is where the graph hits the x-axis. Since the y value is 0 at these points, these are also the x values that make $f(x) = 0$. A real number that makes $f(x) = 0$ is called a **real root** or **real zero**.

expl 3: Use substitution to determine if the values –5 and 3 are zeros of the function $g(x) = 3x^2 + 9x - 30$.

Use the TABLE function on the calculator.

expl 4: Consider the function $f(x) = x(x-3)^2(x+1)$. Algebraically solve $f(x) = 0$ to find the zeros of this function.

What did you get? Notice how these zeros are tied directly to the factors of $f(x)$. The following theorem spells this relationship out.

Zero / Factor Theorem:
Let f be a polynomial function and r be a real number in its domain. The expression $x - r$ is a factor of f if and only if r is a zero of f.

What does "if and only if" mean?

Definition: Multiplicity of a zero: The multiplicity of a zero (or root) is the number of times its corresponding factor appears in the factored form of the polynomial.

The book uses this more exacting definition.

Definition: Real zero of multiplicity m: If $(x-r)^m$ is a factor of a polynomial f and $(x-r)^{m+1}$ is *not* a factor of f, then r is called a real zero of multiplicity m.

expl 5: In the last example, we found the zeros of this function. State each zero's multiplicity.

$f(x) = x(x-3)^2(x+1)$

Zero	Multiplicity
–1	
0	
3	

How many times does each zero's corresponding factor appear?

It turns out that the multiplicity of a zero impacts how the graph appears at that x-intercept. Graph this function on the standard window.

The graph acts differently at the zeroes of –1 and 0 than it does at the zero of 3. Do you see it?

We have this result.

Graphs of Zeroes and Their Multiplicities:

If r is a zero of odd multiplicity, the graph will cross through the x-axis at r.

If r is a zero of even multiplicity, the graph will only touch the x-axis at r.

Odd and even multiplicity simply refers to the number itself. Is it odd or even?

expl 6: Find a polynomial function of degree 3 whose real zeros are -3, 0, and 4. Simplify your answer; leave it in standard form.

How do we use the Zero / Factor Theorem?

expl 7: Find a polynomial function of degree 3 whose real zeros are -3, 0, and 4 and also passes through the point (5, 160). Simplify your answer; leave it in standard form.

Once we get a basic $f(x)$, we want y to be 160 when x is 5. How do we force that?

Turning Points:

Definition: Turning point: A **turning point** in the graph is where it changes from increasing to decreasing or vice versa.

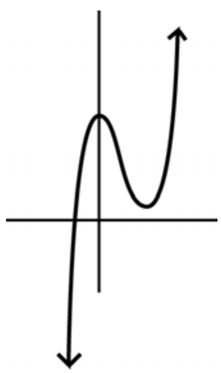
Turning points are explored in calculus.

It happens to be true that a degree n polynomial can have no more than $n-1$ turning points. (Relatedly, the graph can have no more than n zeros.)

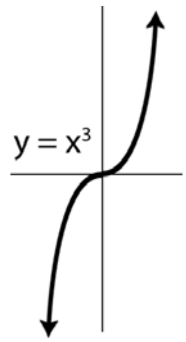
This information is really helpful when you graph by hand.

expl 8: Mark the turning points, if they exist, on the graphs below.

a.)



b.)

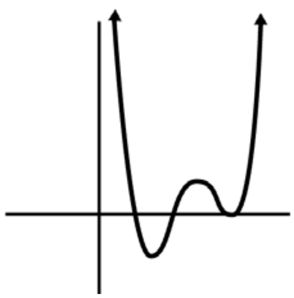


A turning point is where a graph changes from increasing to decreasing or vice versa.

Verifying a Complete Graph:
Knowing what we know (the end behavior, the maximum number of zeros, and the maximum number of turning points that a function can have) helps us graph it by hand or determine if we have a complete graph on the calculator.

For instance, let's say we graph $y = x^4 - 11x^3 + 42x^2 - 64x + 32$. The graph is shown below. Using what you learned from this section, explain how you know it has to be a complete graph.

$y = x^4 - 11x^3 + 42x^2 - 64x + 32$



A complete graph shows all intercepts, turning points, and the end behavior.