How many ways can I reward 4 out of 10 people with a trip to Paris? What if we have four *different* prizes to give out?

General Education Mathematics
Class Notes
Counting: Permutations and Combinations (Section 12.3)

Here, we will see how **permutations** can be used to shortcut the work in some Fundamental Counting Principle (FCP) problems. We will also see another group of problems that will use the related concept of **combinations**.

Let's revisit one of my favorite problems to see how permutations will come into play.

expl 1: Let's say there are ten people (named A, B, C, D, E, F, G, H, I, and J) waiting at a theatre but only the first four in line will get tickets. How many ways can we assign tickets? Fill in the spaces below.

Write in dummy results on top of lines.

________ ________ ________ ________

1st in line   2nd in line   3rd in line   4th in line

Write down how many possibilities for each place in line.

We saw, in the last section, that this can be done with the FCP by calculating $10 \cdot 9 \cdot 8 \cdot 7$ and we get 5,040 ways.

Notice, this is equivalent to finding 10! and dividing by 6! because of how common factors on the top and bottom of a fraction cancel out. See this below.

$$\frac{10!}{6!} = \frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{6\cdot5\cdot4\cdot3\cdot2\cdot1} = \frac{10\cdot9\cdot8\cdot7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 10\cdot9\cdot8\cdot7$$

This is the same as what the FCP gets us.

Recall: Definition: Factorial: Factorials are a quick way to write the product of any non-negative integer and all the positive integers less than it. For instance, $10! = 10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1$. (If it comes up, we define 0! to be 1.)

This brings us to our tool for finding how many ways we can line up four of these ten people in order.

Permutations of n things taken r at a time:
The number of ways we can arrange r items chosen from n total, distinct items is given by $_nP_r = P(n,r) = \dfrac{n!}{(n-r)!}$. Here, we say that the order matters. For instance, the line-up GBCE is different than the line-up BCEG (for example 1, above). Also, as in the theatre examples, repetition of items is *not* allowed.

A person *cannot* occupy two spaces in line.

expl 2a: How many five-letter sequences can I make out of the 26 letters in the alphabet if I *cannot* have repeated letters? Use permutations.

expl 2b: If repeated letters were allowed, I could *not* use permutations. Why not?

expl 3: By hand, calculate the following.

a.) $P(10, 2)$

b.) $P(15, 15)$

c.) $P(10, 8)$

d.) $P(15, 0)$

What does part *d* even mean?

Let's change the scenario up a bit.

expl 4: Let's say we have those ten people (A, B, C, D, E, F, G, H, I, and J) in a room. We want to select four of them to receive a prize. It does *not* matter what order the people are chosen; all four will get the same prize. Write down several possible groups of four people.

Would GBCE be different from BCEG?

How many ways can we do this? Permutations will *not* help us here as the order of the four people does *not* matter. In fact, permutations will *overcount* the number we are after. Instead, we use combinations.

Combinations of n things taken r at a time:
The number of ways we can group r items chosen from n total, distinct items is given by

$${}_nC_r = C(n,r) = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}.$$

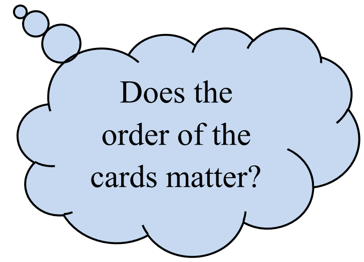
We take $P(n, r)$ and divide by the number of ways to arrange those r things.

Here, we say that the order does *not* matter. For instance, the line-up GBCE is *not* different than the line-up BCEG. Also, as in the theatre examples, repetition of items is *not* allowed.

A person *cannot* occupy two spaces in a group.

Worksheet: Permutations and Combinations:
This worksheet works on these two similar but different concepts and how they are related. That discussion is followed by some practice problems.

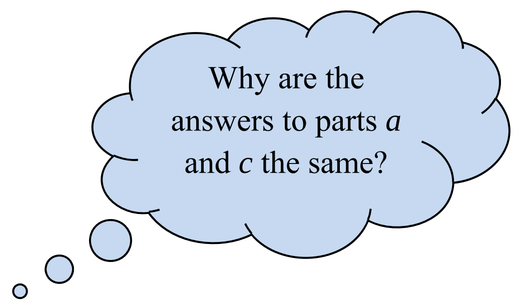
expl 5: In the game of poker, five cards are drawn from a standard 52-card deck making up a "hand". How many different poker hands are possible?



expl 6: By hand, calculate the following.

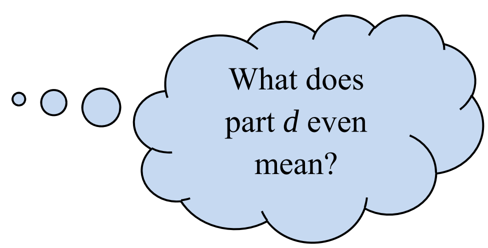
a.) C(10, 2)

b.) C(15, 15)



c.) C(10, 8)

d.) C(15, 0)



Finding Permutations and Combinations on the TI Calculators:

To find $C(10, 2)$ from example 6*a*, we *first* enter the 10 on the home screen. Then press the **MATH** button, and right arrow over to **PRB**. Option 3 should be **nCr**; choose it and it will be put on the home screen after the 10. Then enter 2 and press **ENTER**.

To find $P(10, 2)$ from example 3*a*, follow the same procedure but select option 2 which should be **nPr**.

Try this now!

expl 7: Determine if permutations or combinations are required. Write the number of ways in $P(n, r)$ or $C(n, r)$ notation; then use the calculator to find the number.

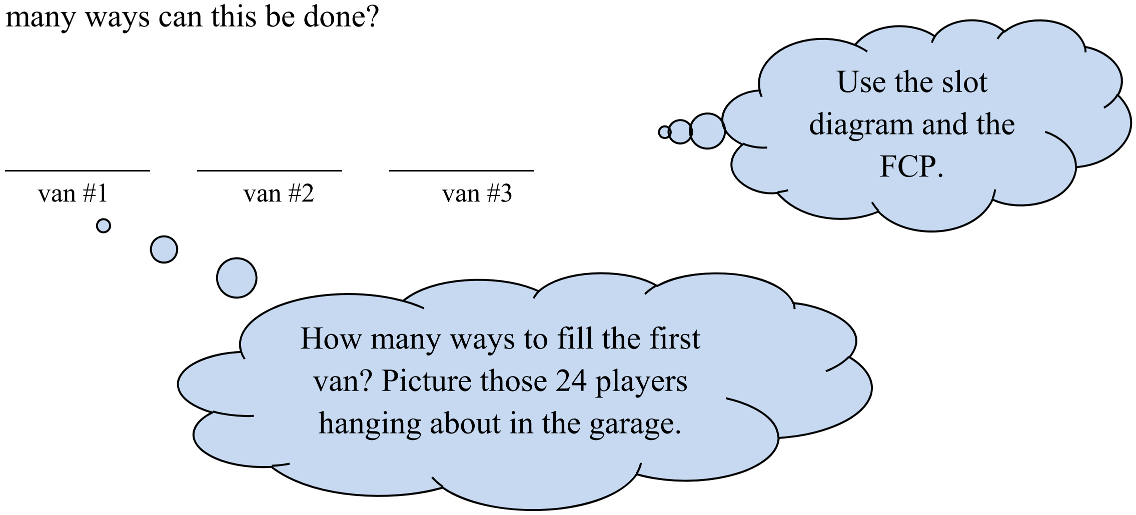
a.) Novak Djokovic, Rafael Nadal, Roger Federer, and Andy Murray are all competing in the Malaysian Open (tennis). How many ways can these four players be seeded in the top four slots in the tournament?

b.) Forty high-school basketball players are competing to be selected for a special training session with LeBron James. Ten lucky players will be selected. How many ways can these ten players be chosen?

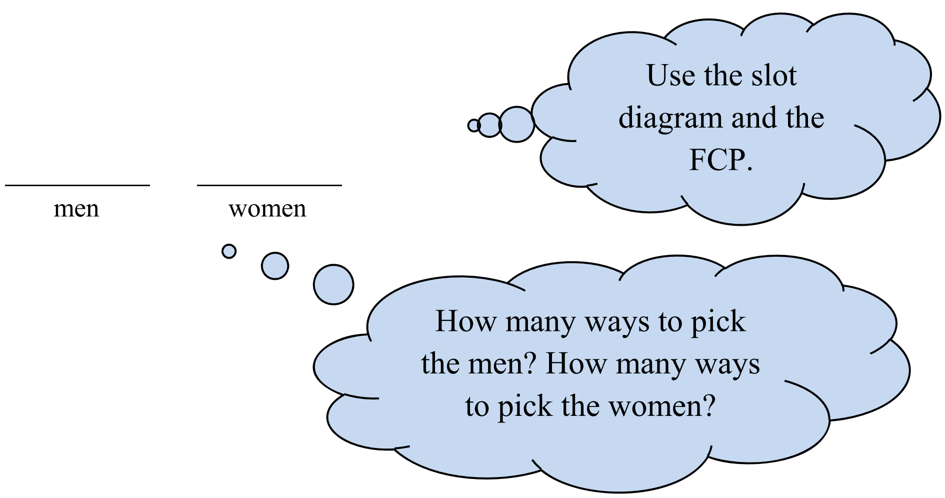
Combining Methods:

Of course, some problems are *not* so cut and dry. Here, we see problems where we need to combine these ideas, usually using the FCP too.

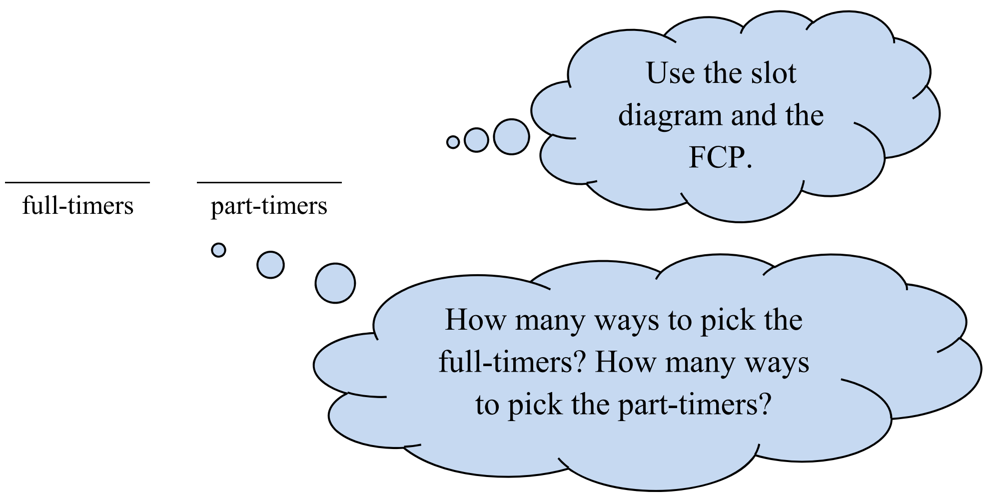
exp 8: There are 24 baseball players that need to fit into three vans for an “away” game. Each van will hold eight players. The order in which the players sit within a van does *not* matter. How many ways can this be done?



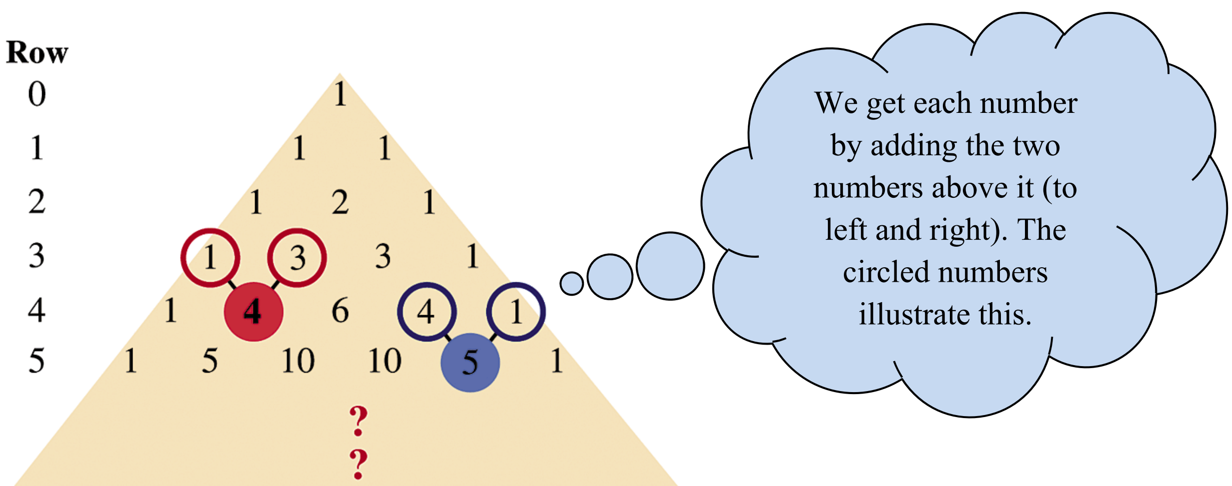
expl 9: The division of student services at your school is selecting two men and two women to attend a leadership conference in Honolulu, Hawaii. If ten men and nine women are qualified for the conference, in how many different ways can management make its decision?



expl 10: A company has 48 full-time employees and 15 part-time employees. They will send a contingent of employees to a conference. From the part-timers, they will select 5 employees. From the full-timers, they will select 3 employees. However, these full-timers will be given the specific jobs of travel coordinator, food coordinator, and lodging coordinator. How many ways can these 8 employees be chosen for the conference?



Pascal's Triangle (or Yang Hui's Triangle if you're Chinese):
This crazy thing pops up in a lot of places. Here is a partial picture.



Fill in the next two rows of Yang Hui's Triangle. Notice each row starts and ends with 1.

What does Pascal's Triangle have to do with Counting?

We have two related patterns to discuss. Let's explore the subsets of the set S = {a, b, c, d} which has four elements.

Size of Subset	Possible Subsets	Number of Subsets
0	∅	1
1	{a} , {b} , {c} , {d}	4
2		
3		
4	{a, b, c, d}	1

I have filled in the subsets of size 0, 1, and 4. Find and record all subsets of sizes 2 and 3 in column 2. Then count the number of subsets and record that in column 3.

Do you recognize these numbers? Where do we find them on Pascal's Triangle?

Now, think about how we made those subsets. We were choosing some number of a total of 4 objects and grouping them. Isn't that the number of combinations of 4 things taken 0, 1, 2, 3, or 4 at a time. Hence, we have these two results.

Pascal's Triangle Counts the Subsets of a Set:
The n^{th} row of Pascal's triangle counts the subsets of various sizes (0, 1, 2, … n) of an n-element set.

Entries of Pascal's Triangle are $C(n, r)$:
The r^{th} entry of the n^{th} row of Pascal's triangle is $C(n, r)$.

expl 11: Where in Pascal's Triangle would we find $C(8, 4)$? In terms of subsets, what does $C(8, 4)$ count?