When can we say the statement $\sim p \vee \sim q$ is true? What does that even mean?

General Education Mathematics
Class Notes
Logic: Truth Tables (Section 3.2)

Truth tables will help us determine whether a compound statement is true or false. We will also use them to determine if two different-looking logical forms (like $\sim(p \wedge q)$ versus $\sim p \vee \sim q$) are the same.

We will complete truth tables for three of the five connectives. And, do you remember DeMorgan's Laws from Set Theory? Well, we have their Logic Theory equivalents here.

Truth Table for Negations:

Let's start off with the simplest connective, negation. Consider some statement p and its negation $\sim p$.

Remember, read $\sim p$ as "*not* p".

p: I like the rain.
$\sim p$: I don't like the rain.

There are two possibilities for the statement p, true or false. We start our table with those values in a column marked p.

$\boldsymbol{p}$	$\boldsymbol{\sim p}$
T	
F	

We will use T for true and F for false.

Now, we read the table across the first row, filling in the second column. We say to ourselves, "If p is true, then is $\sim p$ true or false?" Go ahead and fill in the upper-right cell with T or F.

Now, go to the second row and ask yourself, "If p is false, then is $\sim p$ true or false?" Fill in the lower-right cell with T or F.

Now, we have a complete truth table for the negation of a statement.

This might feel weird because of the abstract arguments. To make it concrete, think of p as a simple statement like "Bears poop in the woods."

If that is true, what is its negation and is it true or false?

Truth Table for Conjunctions:

Recall, the conjunction uses the word "and" to connect two simple statements. Here are some examples.

Bob is fat and Bill is skinny.

Books are fun to read and TV sucks my soul out.

Halloween and Ramadan are holidays.

For these statements to be true, *both* parts must be true.

These conjunctions contain two simple statements we'll denote with p and q. Let's draw out a truth table for the conjunction $p \wedge q$. We need columns for both p and q and a column for $p \wedge q$. I have started filling in the columns for p and q; please complete these columns.

p	q	$p \wedge q$
T	T	
T	F	

What are the four possibilities for the truth values for p and q?

Now, to fill in the third column, we need to work our way along each row. The first row reads that p and q are both true. So, would the statement $p \wedge q$ be true as well, or false? Go ahead and fill in the upper-right cell with T or F.

Looking at each row in turn, ask yourself, "If p and q have the truth values in the row, would the statement $p \wedge q$ be true or false?" Fill in those table cells with the appropriate T or F.

expl 1: Use the truth table above to determine in which row this conjunction would be and then determine if it is true or false.
Buildings are usually made of paper and trees provide us with wood.

Underline each statement and label it as T or F.

Truth Table for Disjunctions:

Recall, the disjunction uses the word "or" to connect two simple statements. Here are some examples.

Bob is fat or Bill is skinny.

Books are fun to read or TV sucks my soul out.

Halloween or Ramadan are holidays.

Recall, "or" means one or the other, *or possibly both*. This is called the "inclusive or", as opposed to the normal English usage ("exclusive").

Again, we see these two simple statements as p and q and we will make up a table. First, fill in the four possibilities for p and q.

p	q	$p \vee q$

Notice, the columns for p and q do *not* change.

Now, to fill in the third column, we need to work our way along each row. Looking at each row in turn, ask yourself, "If p and q have the truth values in the row, would the statement $p \vee q$ be true or false?" Fill in those table cells with the appropriate T or F.

expl 2: Use the truth table above to determine in which row this disjunction would be and then determine if it is true or false.
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The row will depend on how you laid out your table.

Truth Tables for the Conditional and Biconditional:
These will be covered in the next section.

It is helpful to see the similarity between $\vee$ and $\cup$ from Set Theory (both mean OR). Also, notice $\wedge$ and $\cap$ look alike.

Truth Tables for Miscellaneous Statements:

Now that we have worked with truth tables a bit, let's look at some more complicated statements and their truth tables. We should be able to do this with any crazy statement like $p \vee \sim q$ or $\sim(p \wedge q)$ or $(q \wedge r) \vee (r \wedge \sim p)$ or even $\sim(\sim p \vee \sim q) \wedge \sim(p \vee \sim q)$.

expl 3: Write out the truth table for the statement $p \vee \sim q$. Notice the columns we have in this table. These columns will help us go step by step.

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{\sim q}$	$\boldsymbol{p \vee \sim q}$

Why are we using these column headings?

Once we fill in the four possibilities for the truth values for p and q, we really get to work. Fill in those first two columns if you have not.

Considering that q and $\sim q$ have opposite truth values, fill in the third column.

Lastly, a disjunction (like $p \vee \sim q$) will be true if either (or possibly both) parts are true. Complete the truth values for the final column.

DeMorgan's Laws:

Do you remember DeMorgan's Laws from Set Theory? He apparently worked his magic with Logic Theory too and came up with some strikingly similar laws. They are here for your pleasure.

If p and q are statements, then

a.) $\sim (p \wedge q)$ is logically equivalent to $\sim p \vee \sim q$.

b.) $\sim (p \vee q)$ is logically equivalent to $\sim p \wedge \sim q$.

Proving the Logical Equivalence of Statements:

expl 4: Let's use a truth table to prove the first part of DeMorgan's Laws, that $\sim (p \wedge q)$ is logically equivalent to $\sim p \vee \sim q$.

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$p \wedge q$	$\sim (p \wedge q)$
T	T					
T	F					
F	T					
F	F					

To prove the equivalence of two statements (which have the same variables), we will compare the statements' columns. (In our table, these are columns five and seven. Do you see that?) If the columns are identical, then we conclude the statements are equivalent. Complete the table to show these two statements are the same.

expl 5: Recognizing DeMorgan's Laws in common language can help us out. Write the negation of this statement. (Rewriting for purely grammatical reasons is *not* recommended.)
Bob is fat or Bill is skinny.

Write this in symbols, negate it using DeMorgan's Laws, and then rewrite it in words.

expl 6a: Consider the following help-wanted ad. Which applicants listed below should be considered for the position? Explain.
"Management trainee wanted. Applicant must have four-year degree in accounting or three years of experience working in a financial institution."

Applicants: a) Aya has a four-year degree in accounting and has worked two years for a loan company.

b) Heidi has studied accounting in college (but did not graduate) and has worked for five years selling electronics.

c) Monte earned a four-year degree in accounting and has five years of experience working for a credit card company.

expl 6b: You can also see these applicants in the truth table for the disjunction $p \vee q$. I have copied it below. Label the rows which describe each applicant. Take p to be "applicant must have four-year degree in accounting" and q to be "applicant has three years of experience working in a financial institution".

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition: If the final column of a truth table contains all T's, then the statement is always true. Such a statement is called a **tautology**.

Big, Old Truth Tables:

Ooh! Three variables. How many rows do we need? For k variables, we need 2^k rows.

expl 7: Make a truth table for $(q \wedge r) \vee (r \wedge \sim p)$.

p	q	r	$\sim p$	$q \wedge r$	$r \wedge \sim p$	$(q \wedge r) \vee (r \wedge \sim p)$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

Step by step, build the pieces of the statement.

expl 8: Determine if $(q \wedge r) \vee (r \wedge \sim p)$ and $(r \wedge q) \vee \sim(\sim r \vee p)$ are equivalent statements. What other columns do we need?

p	q	r					$(r \wedge q) \vee \sim(\sim r \vee p)$
T	T	T					
T	T	F					
T	F	T					
T	F	F					
F	T	T					
F	T	F					
F	F	T					
F	F	F					

Alternative Method for Constructing Truth Tables:

Of course, different logicians have different methods. Let's use the next example to show off an alternative method to making the tables we have seen so far.

expl 9a: Form a truth table for the statement $(p \vee r) \wedge (p \wedge \sim q)$.

p	q	r	$(p$	$\vee$	$r)$	$\wedge$	$(p$	$\wedge$	$\sim q)$
T	T	T							
T	T	F							
T	F	T							
T	F	F							
F	T	T							
F	T	F							
F	F	T							
F	F	F							

Do you see the table's organization and use of shorthand? It has us copy columns where we need them.

You do *not* fill in the columns from left to right. Column 7 is our "final column".

expl 9b: So, is the statement $(p \vee r) \wedge (p \wedge \sim q)$ true if we know p and q are true but r is false? Which row in the table gives us the answer? Circle the cell in the table with our answer.

Three-Valued Logic:

In real life, statements may *not* be true or false. Some statements are a “definite maybe”. The Polish mathematician Jan Lukasiewicz (1878 – 1956) invented three-valued logic. In addition to the T and F we have seen, we will also see M for maybe.

There are a few rules for this logic. Notice the use of the connectives, *and*, *or*, and *not*.

1. A “true” statement *and* a “maybe” statement is a “maybe” statement. We will denote this as $T \wedge M = M$.

2. A “*not* maybe” statement is a “maybe” statement. In symbols, $\sim M = M$.

3. A “false” statement *and* a “maybe” statement is a “false” statement. In symbols, $F \wedge M = F$.

4. A “true” statement *or* a “maybe” statement is a “true” statement. In symbols, $T \vee M = T$.

5. A “false” statement *or* a “maybe” statement is a “maybe” statement. In symbols, $F \vee M = M$.

6. A “maybe” statement *and/or* a “maybe” statement is a “maybe” statement. In symbols, $M \vee M = M$ and $M \wedge M = M$.

I will get an A in math, maybe.
I have a dog, false.

I will get an A in math *and* I have a dog, false.

I will get an A in math *or* I have a dog, maybe.

expl 10: Assume p is true, q is false, and r is maybe. Find the truth value for $\sim r \wedge (p \vee q)$.

You probably do *not* need a full table.