Roll two dice. What is the probability that you get a sum of 10? Or snake eyes?

General Education Mathematics
Class Notes
Probability: The Basics of Probability Theory (Section 13.1)

Consider rolling two dice. How likely is it that you get two even numbers? Or a sum of seven? How likely is it that a family of three children has exactly two boys? What is the likelihood of being dealt four Aces in a five-card poker hand? We will study these and other probability questions.

Definition: Probability is a measure of the likelihood of a random phenomenon or chance behavior occurring.

What is the probability of pulling an Ace from a poker deck?

Probability deals with experiments that yield random short-term results or **outcomes**, yet reveal long-term predictability. The long-term proportion in which a certain outcome is observed is the **probability** of that outcome.

We will see a **Very Useful Formula** but first, let's define some terminology.

Definition: In probability, an **experiment** is any process that can be repeated in which the results are uncertain. Perhaps, that is rolling a die or tossing a coin. Each time we do the experiment counts as a **trial**.

Perhaps I will pull many cards out of a deck, recording and replacing them as I go.

Definition: The **sample space,** S, of a probability experiment is the collection of all possible outcomes. We will use set notation with the { and } brackets.

What are the many possibilities when you pull a card?

Definition: An **event** is any collection of outcomes from a probability experiment. An event may consist of one outcome or more than one outcome. You can think of an event as a subset of the sample space. Examples are rolling a six on a die or getting two heads when tossing two coins. In general, events are denoted using capital letters such as E. The probability of event E is denoted by $\boldsymbol{P(E)}$.

Definition: A **success** is considered to be an event we are interested in.

Our success is getting an Ace. We will find P(Ace).

expl 1: Determine a sample space for each experiment.

> What are the possible outcomes?

a) We select an iPhone from a production line and determine whether or not it is defective.

b) We select one card from a standard 52-card deck.

c.) We select an Ace from the four Aces of a poker deck and select one of two colored balls (red and blue).

> A poker deck contains four suits: diamonds, hearts, spades, and clubs. The diamonds and hearts are red and the spades and clubs are black. Each suit has thirteen cards: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King. This makes a total of 52 cards. A face card will be defined to be a Jack, Queen, or King. An even number means 2, 4, 6, 8, or 10.

A Very Useful Formula: Probability is the proportion (or fraction) that compares the number of successes to the number of possibilities. *This assumes that the possibilities are equally likely.*

$$\text{Probability of an event} = \frac{\text{number of successes}}{\text{number of total possibilities}}$$

> Sometimes we use percent form.

expl 2: Consider the probability experiment of having three children and recording their sexes.
a.) Identify the outcomes of the probability experiment and write them in set notation, labeled as the sample space, S.

b.) Define the event E = "have exactly one boy". For your sample space above, circle those outcomes that comprise this "event". These will be considered "successes".

c.) Use our Very Useful Formula above to find $P(E)$.

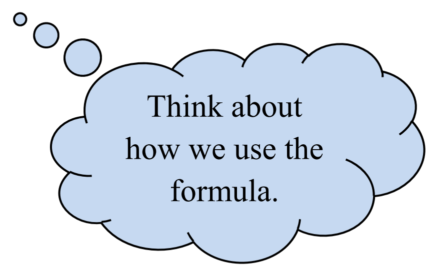
Now considering our Very Useful Formula, we can deduce some **Basic Properties of Probabilities**.

$$\text{Probability of an event} = \frac{\text{number of successes}}{\text{number of total possibilities}}$$

Let's think through them together.

a.) Can the probability of an event be negative? Explain.

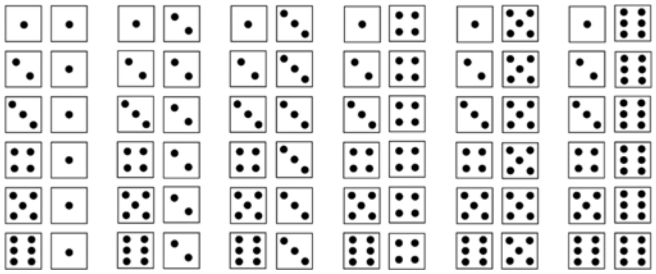
b.) Can the probability of an event be more than 1 (or 100%)? Explain.



c.) What is the sum of the probabilities of all possible outcomes? We will look at a specific example and generalize. Here are the possible number of boys out of three births and their probabilities. What is the sum and why?

Number of boys	**Probability**
0	1/8
1	3/8
2	3/8
3	1/8
Total	

expl 3: We will roll two dice and observe the pair of numbers showing on the top faces. Here is the sample space. Do you agree that these are the only possibilities?

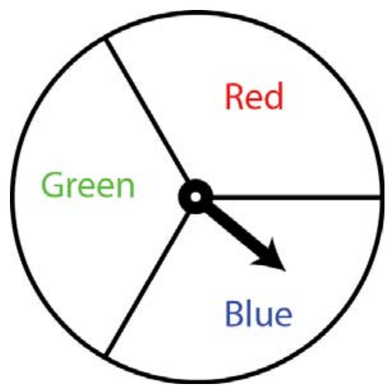


How many successes out of the total possibilities?

How many of the outcomes sum to 10? Circle them. What is the probability (likelihood) of rolling two dice that sum to 10? Use our Very Useful Formula from above.

expl 4: Consider the spinner shown on the right. We will spin this spinner twice, recording the colors in order.

a.) Write out the sample space for this experiment. (Perhaps, imagine the tree diagram.)



b.) Use set notation to write out the event "green does *not* appear in two spins". In other words, which outcomes are within this event? What is the probability of this happening?

c.) Let's say we did this experiment 20 times and recorded our results to the right. *Using this data*, how would you find the probability that this spinner is spun twice and *no* green shows up?

GB, RB, GG, GB, BR,
BG, GR, BG, RR, RG,
RG, GG, GR, BB, BR,
GB, RB, GB, GG, GG

Is it okay for the probabilities of parts *b* and *c* to *not* match?

Types of Probabilities:
There are actually three distinct types of probability.

Theoretical probability was used when we looked at the probability of having exactly one boy with three births. **We say that the outcomes are equally likely**. This means that each outcome (remember these were BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG) are all equally likely to occur as any other. The Very Useful Formula on page 2 uses this model. We may use counting techniques to figure the number of successes and possibilities.

Experimental probability was used when we actually spun our spinner in example 4*c*. We actually do the experiment (or simulate it) and simply count the number of successes. The formula really is the same but some people like to rewrite it like this.

$$\text{Probability of an event} = \frac{\text{number of successes}}{\text{number of trials}}$$

The book words this slightly differently.

Then there are **Subjective Probabilities**. You see this when a weather forecaster tells you that there is a 35% chance of rain tomorrow. Horse betting, another example, is based on probabilities that a certain horse will win. These probabilities are determined by personal judgement. They are *not* based on a probability experiment or counting equally likely outcomes.

Let's do a problem that requires we use our counting skills along with our probability ideas.

expl 5: What is the probability that you are dealt a poker hand with four Aces? We will work it step-by-step using theoretical probability.

a.) How many possible poker hands are there? Remember this involves selecting 5 out of 52 cards. Do we need permutations or combinations?

b.) How many of these hands have four Aces? In other words, assume those Aces are in your hand and now figure how many ways we could pick the fifth card.

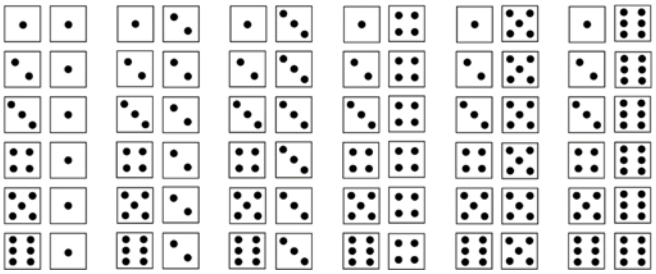
This is the number of successes.

c.) Divide to find the probability of being dealt a "Four-of-a-kind" with Aces.

Definitions: If an event is **impossible**, the probability of the event is 0. If an event is **certain,** the probability of the event is 1.

Think about our Very Useful Formula to make sense of this.

expl 6: Consider the experiment of rolling two dice. Here is the sample space.



Give an event that has 0 probability and an event that has a probability of 1.

expl 7: The data below show the marital status of males and females 15 years old or older in the US in 2013. Answer the following questions.

Marital Status	**Males (in millions)**	**Females (in millions)**
Never married	41.6	36.9
Married	64.4	63.1
Widowed	3.1	11.2
Divorced	11.0	14.4
Separated	2.4	3.2
Total	**122.5**	**128.8**

(Source: U.S. Census Bureau, Current Population Reports)

a.) What is the total number of males?

How many successes? How many possibilities?

b.) If we select a male at random, what is the probability that he is widowed? Round to three decimal places.

c.) If we select a male at random, what is the probability that he is divorced or separated? Round to three decimal places.

Definition: Odds For and **Odds Against:** Probability tells us the ratio of the number of successes to the number of possibilities. Right? **Odds**, on the other hand, measures the number of successes against the number of failures. We have these two formulas which we will explore.

Odds *for* an event = Number of successes / Number of failures

Odds *against* an event = Number of failures / Number of successes

This assumes the outcomes are equally likely.

We will often see odds written with a colon or the word "to" instead of a fraction bar. For instance, the odds of a horse winning a race might be said to be 4 to 1 or 4:1 instead of 4/1.

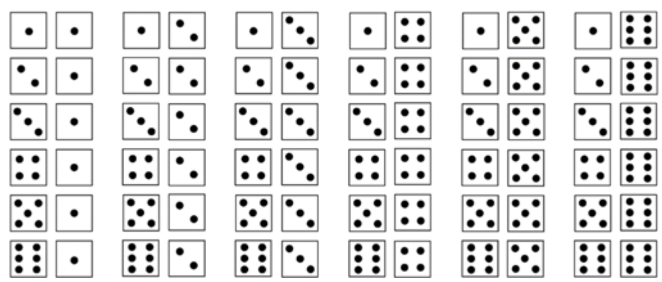
Las Vegas apparently uses the Odds Against an event when it states odds and the book focuses on it. However, you will see both in real life.

expl 8: Consider rolling two dice as we have seen before. Again, the sample space is below.

Notice that the probability that the sum is 7 is 6/36.

Here, we are saying that there are 6 successes and 36 possibilities.

How many failures does that imply?



A failure is an outcome that is *not* a success.

Using the formula above, what are the odds *against* rolling a sum of 7?

Using the formula above, what are the odds *for* rolling a sum of 7?

Reduce odds like you would a fraction.

Genetics:

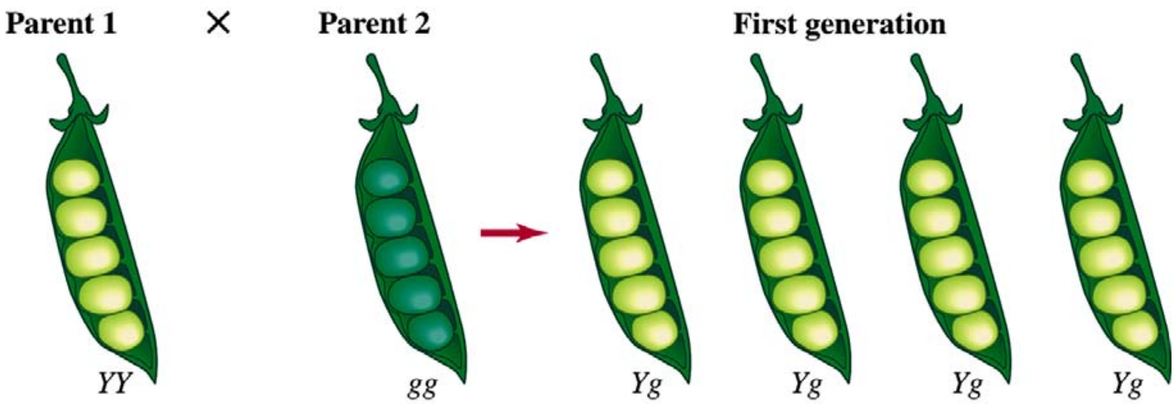
Gregor Mendel is well known for unlocking many mysteries of genetics with his pea experiments. He spent some of his time breeding pea plants that had yellow seeds with pea plants that had green seeds.

He found that when he bred pure yellow-seeded plants with pure green-seeded ones, he always got a yellow-seeded plant. What we now know as recessive and dominant genes are responsible for his results. It turns out that yellow seeds are dominant and green seeds are recessive.

We will explore this through the lens of probability.

Can you think of other dominant or recessive genes?

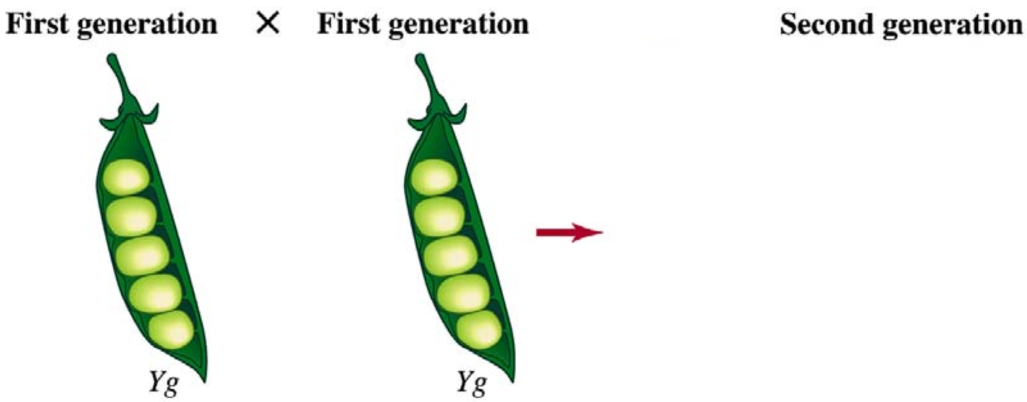
The picture below shows two parent plants and their possible offspring. We use Y to denote the (dominant) gene that produces yellow seeds and g to denote the (recessive) gene that produces green seeds. Notice the parent plants on the left are denoted as YY and gg.



Each offspring will have one *Y* gene from parent 1 and one *g* gene from parent 2, and each will have yellow seeds.

An offspring will receive one gene (that determines the color of the seeds) from each parent plant. If both of these genes are recessive (*gg*, and so the offspring does *not* have the dominant gene), then the plant will have the recessive trait (green seeds). If one or both of the genes is a dominant one (*YY* or *Yg*), then the plant will have the dominant trait (yellow seeds).

But what happens in the next generation, when both *parents* have *Yg* genes? Fill in the four possibilities. What will the colors of their seeds be in each case?



Each offspring will have either a *Y* gene or a *g* gene from each parent. Only the seeds of the offspring with the *gg* pair of genes will be green.

Notice how the theoretical probability that the seeds are green (recessive gene) is ¼ or 0.25 in decimal form.

expl 9: Let's say that smooth versus wrinkled seeds works the same way as yellow versus green seeds. All of the plants in a certain experiment's first generation have smooth (dominant) seeds. The second generation has 5,324 smooth-seeded plants and 1,623 wrinkled-seeded plants. What is the probability that a second-generation plant has wrinkled seeds? Is this consistent with the theoretical results expected from cross breeding?

Where did I put my Very Useful Formula?

The theoretical result is that ¼ of the plants should have 2 copies of the recessive gene, and therefore present that trait.