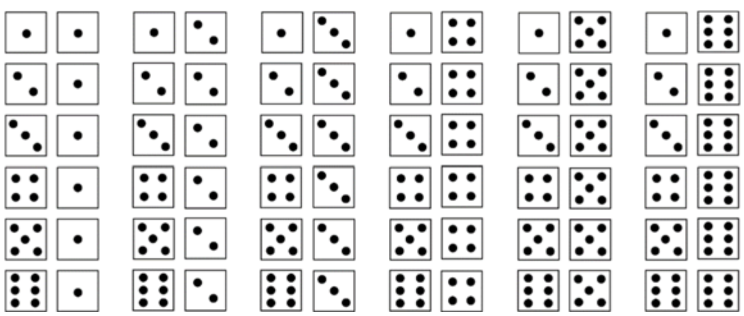
Pick a card from a poker deck. What is the probability it is a Queen *or* a red card?

General Education Mathematics
Class Notes
Probability: Complements and the Union of Events (Section 13.2)

How do we figure the probability of an event *not* occurring? How do we figure the probability of events A *or* B? Does it matter if the events share outcomes in common? Let's jump in and try some out, developing techniques as we go. First, we'll review the mathematical definition of *or*.

What does OR mean?
This may seem obvious but we use OR in math a little differently than in English. When we talk about event E OR event F, we mean either E or F occurs, or *possibly both* occur.

expl 1: Consider rolling two distinguishable, fair, six-sided dice. The sample space is given below. Answer the following questions. This will help us understand an important formula we will see later.



Recall the probability of an event is the number of successes divided by the number of possibilities.

a.) What is the probability of getting a sum of 5? Circle the successes above.

b.) What is the probability of getting a sum of 7? Circle the successes above.

c.) What is the probability of getting a sum of 5 OR a sum of 7? Notice how you have already circled all of the successes.

d.) Can the events "sum of 5" and "sum of 7" happen at the same time (in one roll of the dice)?

Definition: Two events are **mutually exclusive** if they have *no* outcomes in common. This means that they *cannot occur at the same time*. These are also called **disjoint** events.

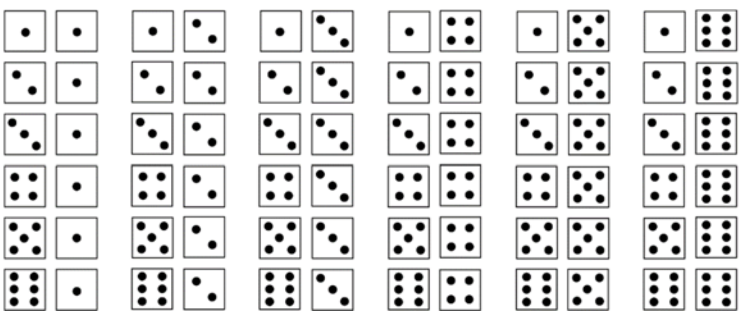
This is true of the events "sum of 5" and "sum of 7". We have a very important rule that applies to mutually exclusive events, which we saw at work in example 1.

Addition Rule for Mutually Exclusive Events:
If E and F are mutually exclusive (or disjoint) events, then $P(E \text{ or } F) = P(E) + P(F)$.

Again, why are "sum of 5" and "sum of 7" considered disjoint?

You will see this written as $P(E \cup F)$. This is the **union** symbol.

expl 2: Again, consider the two dice as before. Answer the following questions.



a.) What is the probability of both dice coming up even, that is P(both even)? Circle the successes above. Do *not* reduce.

b.) What is the probability of getting a sum of 6, that is P(sum is 6)? Circle the successes above. Do *not* reduce.

c.) What is the probability of both dice coming up even OR a sum of 6? Recall, this means that one, or the other, or possibly both events occur. Use the Very Useful Formula for probability. Do *not* reduce.

Number of successes divided by number of possibilities.

expl 2 continued:

d.) If we simply add P(sum is 6) + P(both even), we do *not* get 12/36. What do we get and why is this *not* equal to the probability we are after in part *c*? In other words, what are we counting that we should *not* be?

e.) To further clarify, the outcomes that were overcounted in part *d* satisfy "sum is 6" AND "both even"? In other words, those outcomes satisfy *both* of the events.
Calculate P(sum is 6 AND both even). Do *not* reduce.

AND means *both* events occur.

f.) Now draw a large Venn diagram, with a rectangle encompassing two *intersecting* circles. Label one circle "sum is 6" and the other "both even". Can you picture the outcomes in each circle? Which outcomes are in the *intersection* of the two circles? For each circled outcome in the sample space, place it within the appropriate region of your Venn diagram.

When we add the probabilities as in part *d*, we double-count the outcomes in this intersection.

This overcount needs to be corrected by subtracting those outcomes here.

g.) Use the probabilities you have found to verify this equation.
P(sum is 6 OR both even) = P(sum is 6) + P(both even) – P(sum is 6 AND both even)

This leads us to a very important rule.

> You will see this written as $P(E \cap F)$. This is the **intersection** symbol.

The General Addition Rule:
For any two events E and F, we know $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$.

Since counting the possibilities and successes is a large part of calculating probabilities, you will also see these formulas in terms of the *number of outcomes* in events and *not* their probabilities.

This can work for any formula in this section, but I show it here only for the General Addition Rule. For instance, you may see $n(E \text{ or } F) = n(E) + n(F) - n(E \text{ and } F)$, where $n(E)$ means the number of outcomes in event E, etc.

> Same idea, but uses number of outcomes instead of probabilities.

Worksheet: Probability: Addition Rule:
This worksheet gives us a quick example of the General Addition Rule. We will use experimental probability to explore the rule.

Worksheet: Counting problems involving "OR":
This worksheet explores counting the successes for probability questions involving OR. We will look at when the events do have outcomes in common and when they do *not*. Since we can often use our Very Useful Formula for probability (number of successes divided by number of possibilities), being able to count the successes is vitally important.

expl 3: Assume A and B are events and the following statements are true. Find the missing probability in each case.
a.) P(A$\cup$B) = .85, P(A) = .42, and P(B) = .56. Find P(A$\cap$B).

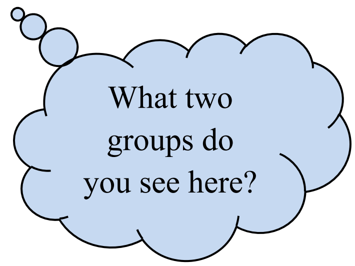
> Algebra skills will help you here.

b.) P(A$\cup$B) = 0.5, P(A$\cap$B) = 0.2, and P(B) = 0.3. Find P(A).

Venn Diagrams:

Venn diagrams can be useful in organizing the information in these problems. Remember that probability is merely the number of successes divided by the number of possibilities.

expl 4: There are 540 people taking part in a medical experiment, in which they either receive the drug being tested or a placebo (a sugar pill that looks like the drug but is not). One hundred people have reported that they are experiencing side effects. Twenty of those 100 were, in fact, given the placebo. Two hundred ten (210) people were given the placebo and are *not* experiencing side effects. If a person is chosen at random, what is the probability that they were given the drug and are *not* experiencing side effects? Use a Venn diagram to organize the information.



Make up another, related probability question for a friend to answer. What is the answer?

Complements:

Recall the complement of a set is the set of all elements that are *not* in the set (but are in the universal set of everything). We use this idea here as well.

Definition: Complement: Let S denote the sample space of a probability experiment and let E denote an event. The complement of E, denoted E', is all outcomes in the sample space S that are *not* outcomes in the event E.

For example 4, the events "person was given the drug" and "person was given the placebo" are complements.

There are other notations used for the complement, like $\bar{E}$, $\sim E$, or E^C.

Consider rolling one six-sided die. Can you think of two events that are complements?

expl 5: I have a bag with ten marbles: four red, three yellow, and three blue.
a.) If I select one marble from the bag, what is the probability that it is blue?

b.) If I select one marble from the bag, what is the probability that it is *not* blue?

How many are *not* blue?

expl 6: I will select a card from *only* the face cards from a poker deck. (The 12 face cards are Jack of Hearts, Queen of Hearts, King of Hearts, Jack of Diamonds, Queen of Diamonds, King of Diamonds, Jack of Spades, Queen of Spades, King of Spades, Jack of Clubs, Queen of Clubs, and King of Clubs. Clubs and Spades are black, Hearts and Diamonds are red.)

a.) If I select one card from my *partial* deck, what is the probability that it is a black Queen?

b.) If I select one card from my *partial* deck, what is the probability that it is *not* a black Queen?

How many cards are *not* black Queens?

These probabilities involving complements can be done by the traditional method of counting the successes and possibilities as we have just seen. However, we could also use a basic rule.

Complement Rule:

If E represents any event and E' represents the complement of E, then $P(E') = 1 - P(E)$.

Since E and E' make up all possibilities, their probabilities must add to 1.

Redo example 5*b* but use this formula. (This assumes you did *not* use the formula when you did the problem originally.)

expl 5*b* again: If I select one marble from the bag (with ten marbles: four red, three yellow, and three blue), what is the probability that it is *not* blue?

Use your answer to part *a*.

expl 7: Consider the experiment of having three children and recording their sexes. Explore the sample space to the right to answer the following questions.

S = {GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB}

a.) What is the probability of having *no* boys?

These events are complements, aren't they?

b.) What is the probability of having *at least one* boy?

expl 8: A store has received a shipment of 15 jackets, 3 of which were labeled "goose-down insulated". As of last Thursday, the store had sold 5 jackets. The next day, they realized that the "goose-down insulated" jackets were mislabeled and were, in fact, insulated with old newspapers. (Wow! That's weird. We will assume the other jackets are labeled correctly.) What is the probability that *at least one* mislabeled jacket was sold? Let's work this step by step.

First, let's find the probability that *no* mislabeled jacket was sold. We will use the complement rule in the end. Do you see why?

a.) How many possible ways could 5 of the total 15 jackets be sold?

Ooh! Combinations!

b.) How many jackets were correctly labeled? How many possible ways could 5 of the correctly-labeled jackets be sold?

c.) So, divide our answers to parts *a* and *b* to find the probability that *no* mislabeled jackets were sold. Round to three decimal places.

d.) The event "*at least one* mislabeled jacket was sold" is the complement of "*no* mislabeled jackets were sold". Use your answer to part *c* to answer the question, "What is the probability that *at least one* mislabeled jacket was sold?"

The Connection Between Complements and Unions:

expl 9: Consider a full poker deck as described to the right. We will select a single card from this 52-card deck. Let's find the probability of that card being neither a face card nor a heart. We will work this step by step.

a.) Draw a Venn diagram with face cards (F) and hearts (H) as the two circles. Place the cards in the appropriate regions. You may abbreviate.

A poker deck contains four suits: diamonds, hearts, spades, and clubs. The diamonds and hearts are red and the spades and clubs are black. Each suit has thirteen cards: Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King. This makes a total of 52 cards. A face card will be defined to be a Jack, Queen, or King. An even number means 2, 4, 6, 8, or 10.

We could answer this question with our Very Useful Formula but choose to explore the connection between complements and unions.

b.) Shade the region of the Venn diagram that shows the event of the selected card "being neither a face card nor a heart". Use the idea of complement to describe this event.

c.) Now, find the probability that the selected card is a face card or a heart. This is $P(F \cup H)$.

d.) Find the probability of the selected card being neither a face card nor a Heart. Use the Complement Rule and your answer to part *c*.

Getting the same answer with two different methods will increase your confidence.