Can't solve it? We will *approximate* a solution.

Differential Equations
Class Notes
Numerical Methods: A Closer Look at Euler's Algorithm (Section 3.6)

After all we have learned, it is still true that the vast majority of differential equations *cannot* be solved by the methods we have studied. Instead, we use numerical methods like we saw earlier with Euler's Approximation Method but, this time around, we will improve on the method.

Recall, Euler's Method was as follows.

Euler's Method Procedure:
For first-order diff. eq. $y' = f(x, y)$ with initial condition (x_0, y_0) and step size h, we use the following formulas.

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h \cdot f\left(x_n, y_n\right) \quad \text{where } n = 0, 1, 2, \ldots$$

You will recall that our goal was to obtain an approximation of the solution $\phi(x)$ to the initial value problem at those points x_n in some interval (a, b). We generate values $y_0, y_1, y_2, \ldots$ that approximate $\phi(x_0), \phi(x_1), \phi(x_2), \ldots$. Truly, we are given $y_0 = \phi(x_0)$ but we approximate the others using the formula above.

Depending on h, we get different approximations for $y_1, y_2, \ldots$. As h gets smaller, the approximation gets more accurate. However, cost and round-off error increases. Can we make a better approximation? You betcha! The book explores this and yields the following replacement formula for y_{n+1}.

Trapezoid Scheme:

$$y_{n+1} = y_n + \frac{h}{2} \cdot \left[f\left(x_n, y_n\right) + f\left(x_{n+1}, y_{n+1}\right) \right] \quad \text{where } n = 0, 1, 2, \ldots$$

Notice that this formula for y_{n+1} actually uses the value for y_{n+1} in its calculation. This makes it an **implicit method**.

To avoid using y_{n+1} in its own formula, we can use Euler's method to estimate y_{n+1} (calling it y^*_{n+1}) and use this estimate in the trapezoid scheme. This is an example of a **predictor-corrector method**. This turns our formula for y_{n+1} into the following, labeled on the next page as the Improved Euler's Method.

Improved Euler's Method:

For first-order diff. eq. $y' = f(x, y)$ with initial condition (x_0, y_0) and step size h, we use the following formulas.

Do you see the *original* Euler's formulas here?

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + \frac{h}{2} \cdot \left[f(x_n, y_n) + f\left(x_n + h,\ y_n + h \cdot f(x_n, y_n)\right) \right] \quad \text{where } n = 0, 1, 2, \ldots$$

So, what do we do with this? The book has provided our steps in the form of a computer subroutine.

Improved Euler's Method Subroutine

Purpose To approximate the solution $\phi(x)$ to the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

for $x_0 \le x \le c$.

INPUT x_0, y_0, c, N (number of steps), PRNTR (= 1 to print a table)

Step 1 Set step size $h = (c - x_0)/N$, $x = x_0$, $y = y_0$

Step 2 For $i = 1$ to N, do Steps 3–5

Step 3 Set

$$F = f(x, y)$$
$$G = f(x + h, y + hF)$$

Step 4 Set

$$x = x + h$$
$$y = y + h(F + G)/2$$

Step 5 If PRNTR = 1, print x, y

Let's go to the next page for our first example. We'll do this first one by hand but then use an online calculator for remaining problems.

expl 1: Use the improved Euler's method subroutine with step size $h = 0.2$ to approximate the solution to the initial value problem $y' = \frac{1}{x}\left(y^2 + y\right), \quad y(1) = 1$, at the points $x = 1.2$, 1.4, 1.6, and 1.8. (Thus, input $N = 4$.) Compare these approximations with those made using Euler's Method as previously seen (exercise 6, section 1.4). Round as little as possible.

Step 4 of the subroutine says "$x = x + h$". In computer language, this means "the new x equals the old x plus h". They dispense with the "n+1" and "n" subscript notation.

We go through the subroutine as a computer would. Fill this table with your approximations as you go.

x	***y***
1	1
1.2	
1.4	
1.6	
1.8	

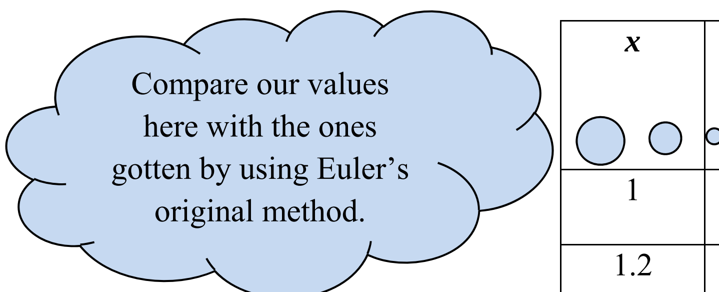
First, get organized (identify $f(x, y)$, x_0, y_0, and h) and define F and G. Do *not* bother to simplify G.

Our first y value is in the table since we are given it. Let's repeatedly perform the subroutine (step 4 really) to find the missing values.

(extra room)

Since we use the previous answers in each calculation, round-off error gets worse as we go. The more you round answers, the worse this error is.

Let's compare our values to the values gotten from the original Euler's method. I have recorded my values for the Improved Euler's Method. They may differ slightly from yours due to rounding.



x	***y* (Improved Euler's Method)**	***y* (Euler's Method)**
1	1	1
1.2	1.48	1.400
1.4	2.25	1.960
1.6	3.66	2.789
1.8	6.91	4.110

Actually, this diff. eq. is separable. (In reality, we were just playing with this Euler method since it was *not* needed.) In fact, and you could verify this, the solution is $\frac{y}{y+1} = \frac{1}{2}|x|$. Let's check how good this improved Euler's Method really did do. Here's what I got for y using the actual solution.

x	***y* (Improved Euler's Method)**	***y* (Actual Solution)**
1	1	1
1.2	1.48	1.5
1.4	2.25	2.33
1.6	3.66	4
1.8	6.91	9

Compare the actual solutions to the approximations. What do you notice as we go down the table?

Doing this process by hand can teach you a lot about it. However, my intent is to have you use an online calculator for homework and exam problems. We will use the online calculator www.math-cs.gordon.edu/~senning/desolver. (This link is available on www.stlmath.com.) You will want to select, as the Method, Heun (Improved Euler).

expl 2: Use the improved Euler's method subroutine with step size $h = 0.1$ to approximate solutions to $y' = 4\cos(x+y)$, $y(0) = 1$, at the points $x = 0, 0.1, 0.2, 0.3, \dots 1.0$. Use your answers to make a rough sketch of the solution on $[0, 1]$.

For the online calculator, input the following.

$f(t, y) = 4*\cos(t+y)$

$t_0 = 0$

$y_0 = 1$

$t_1 = 1.0$

$h = 0.1$

Get organized. The online calculator defines the diff. eq. as $\frac{dy}{dt} = f(t, y)$.

Select "Graph and Data points" as the "Output format". Use the next page to record the results. Remember the first column of output gives us the x-values with the approximated y-values in the second column.

Record the table and draw the graph here.

x	*approx. y*
0	1
0.1	
0.2	
0.3	
0.4	
0.5	
0.6	
0.7	
0.8	
0.9	
1.0	

Euler's Method with Tolerance:

We want to approximate $\phi(c)$ to a desired accuracy, ε (epsilon). We did this when we looked at Euler's method before.

This accuracy depends on h, the step size.

Our strategy: Estimate $\phi(c)$ for a given h, halve h, compute again and again, continuing to halve h each time, until two consecutive $\phi(c)$ estimates differ by less than ε. The final estimate will be taken for $\phi(c)$.

The book also provides a subroutine for this procedure which I have on the next page with our example. However, I do *not* need you to do any of these by hand. Use the online calculator.

Improved Euler's Method With Tolerance

Purpose To approximate the solution to the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

at $x = c$, with tolerance ε

INPUT x_0, y_0, c, ε,
M (maximum number of halvings of step size)

Step 1 Set $z = y_0$, PRNTR $= 0$
Step 2 For $m = 0$ to M, do Steps 3–7[††]
Step 3 Set $N = 2^m$
Step 4 Call IMPROVED EULER'S METHOD SUBROUTINE
Step 5 Print h, y
Step 6 If $|y - z| < \varepsilon$, go to Step 10
Step 7 Set $z = y$
Step 8 Print "$\phi(c)$ is approximately"; y; "but may not be within the tolerance"; ε
Step 9 Go to Step 11
Step 10 Print "$\phi(c)$ is approximately"; y; "with tolerance"; ε
Step 11 STOP

OUTPUT Approximations of the solution to the initial value problem at $x = c$ using 2^m steps

expl 3: Use the improved Euler's method with tolerance to approximate the solution to $y' = 1 - \sin(y)$, $y(0) = 0$, at $x = \pi$. Use $\varepsilon = 0.01$ tolerance. Use a stopping procedure based on absolute error.

Be sure to select the *improved* method. You will have to use 3.1416 for π.

h	***Approximate*** $y(\pi)$
3.1416	3.141612
3.1416 / 2	1.056633
3.1416 / 4	
3.1416 / 8	
3.1416 / 16	

We will start with an h value of 3.1416 (the number of steps (N) being 1), recording the value given for y at $x = \pi$. We then halve this value of h and run the calculator again. Continue like this, recording your values for $y(\pi)$.

I have recorded the first couple of approximations. Is their difference within the tolerance?

What is your approximation for $y(\pi)$?

Revisit Heating and Cooling of Buildings:

We can use this stuff to approximate solutions architects might need as they design a building.

expl 4: In a previous section, we modeled the temperature inside a building by the initial value problem $\frac{dT}{dt} = K\left[M(t) - T(t)\right] + H(t) + U(t), \quad T(t_0) = T_0$, where M is the outside temperature, T is the inside temperature, H is the additional heating rate (people, machines, etc.), U is the furnace and air conditioner heating/cooling rate, K is a positive constant (related to doors, windows, and insulation), and T_0 is the initial temperature at time t_0. In a typical model, $t_0 = 0$ (midnight), $T_0 = 65$°F, $H(t) = 0.1$, $U(t) = 1.5\,(70 - T(t))$, and $M(t) = 75 - 20\cos\left(\pi \cdot t/12\right)$.

The constant K is usually between ¼ and ½, depending on such things as insulation. To study the effects of insulating this building, consider the typical building described above and use the improved Euler's method subroutine with $h = 2/3$ (step size) to approximate the solution to this diff. eq. on the interval $0 \le t \le 24$ (1 day) for $K = 0.2$, 0.4, and 0.6.

First, form dT/dt. This will be $f(t, y)$ in the online calculator. Careful with the variable change and required multiplication signs!

Run the calculator for each value of K.

Again, use 3.1416 for π.

Approximate Temperatures Inside the Building Throughout a 24-Hour Day, ***T*(*t*)**				
time	***M*(*t*)**	***K* = 0.2**	***K* = 0.4**	***K* = 0.6**
midnight ($t = 0$)	55°F	65°F	65°F	65°F
6:00 am ($t = 6$)	75°F			
12:00 noon ($t = 12$)	95°F			
6:00 pm ($t = 18$)	75°F			
midnight ($t = 24$)	55°F			

By the way, which value of K corresponds to the best insulation?