If we know the rate of change in a single population or closed system, we can solve that diff. eq.

Differential Equations
Class Notes
Compartmental Analysis (Section 3.2)

Section 3.1 gives a nice overview of the process of developing a mathematical model for a physical situation. We will work in section 3.2 with what is called a **compartmental system**.

**Handout:** This will be given out in class but is available on www.stlmath.com under Assorted Handouts and Tutorials. It will serve as a good reference as we proceed.

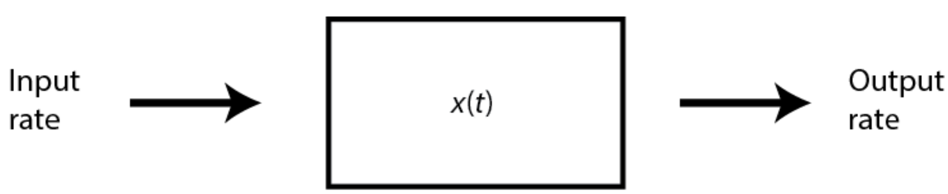
Differential Equations Study Guide (site, integral-table.com)
http://integral-table.com/downloads/ODE-Summary.pdf

Many complicated processes can be broken down into distinct stages and the entire system can be modeled be describing the interactions between these stages. Such systems are **compartmental**, often depicted by a block diagram.

We will study the basic unit of a system, a **compartment**, and analyze processes that can be handled by such a model. There are many types of problems that can be solved this way.

**Mixing Problems:**

Consider a tank with, say, salt water in it. Let's say there is a flow of water (or salt water) entering the left side and a flow of water leaving the tank on the right. We could measure the amount of salt in the tank at any time $t$ and therefore know the concentration of salt in the tank. Now, wouldn't that be fun? Here is a picture.

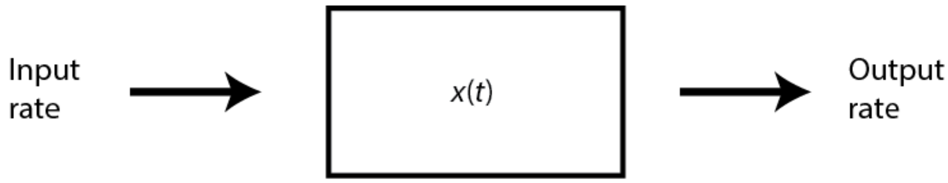



$x(t)$ = amount of salt (kg) in tank at time $t$

The concentration of salt in the tank would be $x(t)$ divided by the volume of the tank. We will use the fact that $\frac{dx}{dt} = \textit{input rate} - \textit{output rate}$. Hello, old friend, my differential equation.

expl 1: A brine solution of salt flows at a constant rate of 8 L/minute into a large tank that initially held 100 L of brine solution in which was dissolved .5 kg of salt. The solution in the tank is kept well stirred and flows out of the tank at the same rate. If the concentration of salt in the brine entering the tank is .05 kg/L, determine the mass of salt in the tank after $t$ minutes. When will the concentration of salt in the tank reach .02 kg/L?

Let's start off by using that picture from above and putting our information in place.




$x(t)$ = amount of salt (kg) in tank at time $t$

Find the input and output rates in kg/minute. Label them on the picture.

Form your equation for *dx/dt* using the generic one at the bottom of page 1.

We know the initial amount of salt, or $x(0)$. What is it? Do you see an initial value problem here? Solve it.

Once you get your diff. eq., you will see it as linear.

(extra room for work)

Once you know $x(t)$, we can answer the question, “When will the concentration of salt in the tank reach .02 kg/L?”

**Population Models:**

How do we predict the growth of a population? A single population can be thought of as a compartment.

Let $p(t)$ be the population of some species at time $t$. There will be a growth (input) rate and a death (output) rate. We see our diff. eq. in the Malthusian Model.

**Malthusian Model for Population Growth (Exponential):**
Again, $p(t)$ is the population at time $t$. We have the following.

$$\frac{dp}{dt} = k_1 p - k_2 p, \qquad p(0) = p_0$$

or $$\frac{dp}{dt} = kp, \qquad p(0) = p_0$$

where $k_1$ is the proportionality constant for growth and $k_2$ is the proportionality constant for death. To simplify, we use $k = k_1 - k_2$. We will assume $k_1 > k_2$ and so $k > 0$.

This equation is separable and solving for $p(t)$ gets us $p(t) = p_0 e^{kt}$.

College algebra students play with this equation.

expl 2: In 1990, the Department of Natural Resources released 1,000 splake (a crossbreed of fish) into a lake. In 1997, the population was estimated to be 3,000 splake. Using the Malthusian law for population growth, estimate the population of splake in the year 2020.

No diff. eq. needed here.

First, find $k$.

expl 3: The initial mass of a certain species of fish is 7 million tons. The mass of fish, if left alone, would increase at a rate proportional to its mass, with a proportionality constant of 2/year. However, commercial fishing removes fish at a rate of 15 million tons per year. When will all the fish be gone? If the fishing rate is changed so that the mass of fish remains constant, what should that rate be?

Let $p(t)$ = mass of fish at time $t$.
We know that $dp/dt$ is the birth rate minus the fishing rate.

What kind of equation is $dp/dt$?

Did you see this as an initial value problem? Use the initial population of 7 million tons to nail down the formula for $p(t)$.

expl 3 (continued): Now, we can answer the questions. When will all the fish be gone? If the fishing rate is changed so that the mass of fish remains constant, what should that rate be?

**Radioactive Decay:**

The amount of a radioactive substance decays in such a way so that the Malthusian model can be used. In fact, the rate of decay is proportional to the amount of substance present much like the rate of population growth we saw earlier. We have the following (which could be derived with our beautiful methods for solving differential equations but will be left for you to explore on your own).

We *would* start by letting $M$ represent the amount of substance present. Then we know $dM/dt = kM$ for some constant $k$. Yada, yada, yada, …

**Radioactive Decay:**
We will let $m(t)$ be the mass of substance at time $t$. After solving the diff. eq. imagined above, we have the following.

$$m(t) = m_0 e^{kt}$$

Here, $k$ is the decay constant that is particular to the substance and $m_0$ is the initial amount.

expl 4: Initially there are 300 grams of a radioactive substance and after 5 years, there are 200 grams remaining. How much time must elapse before only 10 grams remain?

This could be started as far back as solving a diff. eq. but that would only get us the equation for $m(t)$ we are given above. We will simply start there.

First, find $k$.