We will define what exact means and explore how to solve these equations.

Differential Equations
Class Notes
Exact Equations (Section 2.4)

First, let's cover some preliminary definitions.

Definition: Level curves of $F(x, y)$: sets on which the function value is constant. In other words, where $F(x, y) = c$ for some $c \in \mathbb{R}$.

Definition: Total differential of F (or dF) for $F(x, y)$: This is defined to be
$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy .$$

Introductory Explorations:
Consider a function $F(x, y) = c$ for some $c \in \mathbb{R}$. Then $dF = 0$, right?

That implies that $0 = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$. Alternative versions of this diff. eq. are $\frac{\partial F}{\partial x} dx = -\frac{\partial F}{\partial y} dy$ and $\frac{dy}{dx} = \frac{\partial F / \partial x}{-\partial F / \partial y}$. These are three forms of the same diff. eq. that we will define as **exact**.

Here, $F(x, y) = c$ would be **solutions** to these differential equations.

Definition: Exact Differential Form: The differential expression $M(x, y)dx + N(x, y)dy$ is said to be **exact** in a rectangle R if there exists a function $F(x, y)$ such that $\partial F / \partial x = M$ and $\partial F / \partial y = N$ for all (x, y) in the rectangle R.

That is, $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = M \cdot dx + N \cdot dy$.

Here, we use the abbreviation M for $M(x, y)$.

Definition: Exact Differential Equation: If $M(x, y)dx + N(x, y)dy$ is an exact diff. expression, then $M(x, y)dx + N(x, y)dy = 0$ is an **exact diff. eq.**.

We need a test to determine if an equation is exact and a method to solve these equations.

Test for Exactness:

From calculus, we know $\partial/\partial y\left(\partial F/\partial x\right) = \partial/\partial x\left(\partial F/\partial y\right)$ because of the equality of continuous partial derivatives. Using $\partial F/\partial x = M$ and $\partial F/\partial y = N$, we will write this as

$\partial/\partial y\left(M\right) = \partial/\partial x\left(N\right)$.

Using Khan Academy notation, $M_y = N_x$.

Theorem 2: Test for Exactness:
Suppose the first partial derivatives of $M(x, y)$ and $N(x, y)$ are continuous in a rectangle R. Then $M(x, y)dx + N(x, y)dy = 0$ is an exact diff. equation in R if and only if the compatibility condition $\partial/\partial y\left(M\right) = \partial/\partial x\left(N\right)$ holds for all (x, y) in R.

Alternatively,

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y).$$

expl 1: Show this diff. eq. is exact.

$$\left[2x + y\cos(xy)\right]dx + \left[x\cos(xy) - 2y\right]dy = 0$$

Identify M and N. Show that $\partial/\partial y\left(M\right) = \partial/\partial x\left(N\right)$.

$$\frac{d}{dx}\cos(xk) = -k\sin(xk)$$

$$\frac{d}{dy}\cos(ky) = -k\sin(ky)$$

When a Linear Equation Appears Non-Linear:
This was mentioned in the last section of notes but *not* needed in the homework. Now it will be.

expl 2: Check to see if the equation is linear.
$\theta dr + (3r - \theta - 1)d\theta = 0$

> Can it be written in the form $\frac{d\theta}{dr} + P(r)\cdot\theta = Q(r)$ or $\frac{dr}{d\theta} + P(\theta)\cdot r = Q(\theta)$?

> It is a matter of which variable you take to be the dependent one.

Method for Solving Exact Differential Equations:

a.) If $M{\cdot}dx + N{\cdot}dy = 0$ is exact (meaning, if $M_y = N_x$), then $\partial F/\partial x = M$. Integrate $\partial F/\partial x = M$ to get $F(x,y) = \int M\,dx + g(y)$.

> This $g(y)$ is similar to the constant of integration. However, since we integrate M with respect to x, we acknowledge this part may contain y.

b.) To find $g(y)$, take the partial derivative of this with respect to y to get $\frac{\partial}{\partial y}F(x,y) = \frac{\partial}{\partial y}\int M\,dx + g'(y)$.

Since $\frac{\partial}{\partial y}F(x,y) = N$, we solve for $g'(y)$ and get $g'(y) = N - \frac{\partial}{\partial y}\int M\,dx$.

c.) Integrate this $g'(y)$ with respect to y to find $g(y)$ up to a constant (meaning, we will end up with a constant due to integration). Then write $F(x,y) = \int M\,dx + g(y)$.

d.) The solution to the diff. eq. $M{\cdot}dx + N{\cdot}dy = 0$ is then said to be $F(x,y) = c$ for some $c \in \mathbb{R}$.

Alternatively, start with $\partial F/\partial y = N$ and so $F(x,y) = \int N\,dy + h(x)$ and so forth. We do this because it is easier sometimes.

expl 3: Determine if the equation is exact. If so, solve it.

$$\left(2x+y\right)dx+\left(x-2y\right)dy=0$$

Pick out M and N. See if $M_y = N_x$.

Follow steps *a* through *d*.

Initial Value Problems:

expl 4: Solve.

$$\left(\tan y - 2\right)dx + \left(x\sec^2 y + \frac{1}{y}\right)dy = 0, \quad y(0) = 1$$

Pick out M and N. See if $M_y = N_x$.

Follow steps *a* through *d*.

Use $y(0) = 1$ to find c.

Our solution contains $\ln|y|$ which can be taken as $\ln y$ around the point (0, 1) since $y > 0$. The book often simplifies a solution this way. However, MyMathLab may *not*.

Further Exploration: How do we deal with non-exact equations?:

expl 5: Consider the equation $\left(y^2 + 2xy\right)dx - x^2 dy = 0$.

Pick out M and N. Show $M_y \neq N_x$.

a.) Show it is *not* exact.

b.) Show that multiplying by y^{-2} yields a new equation that *is* exact.

c.) Use the solution of the new exact equation to solve the original equation.

d.) Were any solutions lost in the process?

How did multiplying by y^{-2} change the solution?

Can you solve for y explicitly?

Lost solutions?

Since the original equation and our modified one differ only by a factor of y^{-2}, they should have the same solutions except when this factor is undefined or zero, or when $y = 0$. Notice this is a solution to the original diff. eq. that does *not* show up because we divided out by y^2.