A building can be thought of as a "compartment" from which we get some interesting diff. eqs.

Differential Equations
Class Notes
Heating and Cooling of Buildings (Section 3.3)

We want to determine a mathematical model that describes the 24-hour temperature cycle within a building as a function of outside temperature, the heat generated inside the building, and the heat generated by a furnace or air conditioner.

Our main questions are,

1. How long does it take the building temperature to change substantially?
2. How does the building temperature vary *without* a furnace or air conditioner?
3. How does the building temperature vary *with* a furnace or air conditioner?

We will treat the building as a "compartment". Let $T(t)$ be the inside building temperature at time t. Usually, t is the number of hours or minutes after some initial point.

So, $T(t)$ and its rate of change (dT/dt) are dependent on

1. heat produced by people, lights, machines, etc. where $H(t)$ is the rate of increase in temperature caused by this,
2. heat or cool produced by a furnace or air conditioner where $U(t)$ is the rate of increase or decrease in temperature caused by this, and
3. the outside temperature denoted by $M(t)$.

Notice how H and U are rates of change, *not* the heat itself.

Newton's Law of Cooling states the rate of change in the temperature $T(t)$ is proportional to the difference between the outside temperature $M(t)$ and the inside temperature $T(t)$. In other words, we have $\frac{dT}{dt} = K\left[M(t) - T(t)\right]$. Let's add to this the rates of change H and U seen above and we get $\frac{dT}{dt} = K\left[M(t) - T(t)\right] + H(t) + U(t)$ where $K > 0$ depends on the physical properties of the building such as the number of windows and doors and the quality of insulation.

This can be shown to be linear, in the form of $\frac{dT}{dt} + P(t)\cdot T = Q(t)$ where $P(t) = K$ and $Q(t) = KM(t) + H(t) + U(t)$. (Do you see that?) When we solve this diff. eq., we get $T(t) = e^{-Kt}\left[\int e^{Kt}\left[KM(t) + H(t) + U(t)\right]dt + C\right]$. The C is the constant of integration.

Definition: Time constant for building: We will define $1/K$ to be the time constant for the building (without furnace or air conditioning). It is typically 2 - 4 hours but can be shorter or longer depending on if windows are left open, fans are on, building is well-insulated, etc.

expl 1: On a hot Saturday morning while people are working inside, the a-c keeps the temperature inside a building at 24° C. At noon, the a-c is turned off and people go home. The temperature outside is a constant 35° C for the rest of the afternoon. If the time constant for the building is 4 hours, what will the temperature inside the building be at 2:00 p.m.? At 6:00 p.m.? When will the temperature inside the building reach 27° C?

24° C ≈ 75° F
35° C = 95° F

Let t = number of hours past noon and $T(t)$ = inside temperature at time t.

At noon, we have no people nor a-c. So, $U(t) = H(t) = 0$.

We were also given $T(0)$ and $M(t)$. What are they? While you're at it, find K.

Use the formula given at the bottom of page 1 for $T(t)$.

expl 1 (continued): Remember that this is an initial value problem. So, use that to determine that constant C. Once you get a formula for $T(t)$, answer the questions with algebra. Phrase your answers in sentence form.

expl 2: A garage with no heating or cooling has a time constant of 2 hours. If the outside temperature varies as a sine wave with a minimum of 50° F at 2:00 a.m. and a maximum of 80° F at 2:00 p.m., determine the times at which the building reaches its lowest temperature and its highest temperature, assuming the exponential term has died off.

Let t = number of hours past 2:00 a.m. and $T(t)$ = inside temperature at time t.

No heating or cooling and no people means $U(t) = H(t) = 0$.

Average the max. and min. temperatures to find M_0.

$B = \frac{80-50}{2} = 15$

The book provides us with $M(t) = M_0 - B\cos(\omega t)$. Here, $B > 0$ is a constant, M_0 is the *average* outside temperature, and $\omega = 2\pi/24 = \pi/12$ radians per hour.

You'll need this formula: $\int e^{au}\cos(bu)\,du = \frac{e^{au}}{a^2+b^2}\left(a\cos(bu) + b\sin(bu)\right) + c$

(extra room for work)

What it means for the exponential term to die off is that as $t \to \infty$, $Ce^{-t/2} \to 0$. We can remove this term from our $T(t)$ formula to find the max. and min.