If a diff. eq. is *not* exact, we will multiply it by a function $\mu(x, y)$ that will make it exact.

Differential Equations
Class Notes
Special Integrating Factors (Section 2.5)

Theoretically, an integrating factor exists for *all* differential equations in the form $M(x, y)\,dx + N(x, y)\,dy = 0$. However, *no* general rule is known. Methods are known for certain types and tend to be somewhat messier than what we see here.

In this section, we look at a particular case. We will start with a non-exact diff. eq. and transform it into an exact equation. Then we can use the solution method from the previous section.

Definition: Integrating Factor: If the equation $Mdx + Ndy = 0$ is *not* exact but $\mu(x, y)Mdx + \mu(x, y)Ndy = 0$ is exact, then the function $\mu(x, y)$ is called an **integrating factor** of the original equation. Here, M and N are functions in both x and y, or rather $M = M(x, y)$ and $N = N(x, y)$.

We saw this in the last example in the Class Notes for Exact Equations.

Once we determine $\mu(x, y)$, or are given it, we multiply our non-exact equation by it and the equation becomes exact.

But wait! Since we multiply by $\mu(x, y)$, it is possible that we **lose or gain solutions**. We must always check whether any solutions to $\mu(x, y) = 0$ are *truly* solutions to the *original* diff. eq. In addition, we will also find any values that make $\mu(x, y)$ undefined as these may be solutions.

How do we find $\mu(x, y)$?

We have a theorem.

Lost (additional) solutions may exist where $\mu(x, y)$ is undefined. **Extraneous (gained) solutions** may exist when it's zero.

Theorem 3: Special Integrating Factors:
Consider the *non*-exact equation $M(x, y)dx + N(x, y)dy = 0$.

Case 1: If $f = \left(M_y - N_x\right)/N$ depends only on x, then $\mu(x) = e^{\int f\,dx}$ is an integrating factor for the diff. eq.

Case 2: If $f = \left(N_x - M_y\right)/M$ depends only on y, then $\mu(y) = e^{\int f\,dy}$ is an integrating factor for the diff. eq.

Here, I diverge from the book's notation, preferring to use the notation $M_y = \dfrac{\partial M}{\partial y}$ and $N_x = \dfrac{\partial N}{\partial x}$. I also use this construction f to simplify the write-up.

The book contends that f should be continuous but this is *not* mentioned in other resources.

In order to solve the various diff. eqs. that we are given, we must be able to categorize them.

expl 1: Identify the diff. eqs. as separable, linear, exact, or having an integrating factor that is a function of either x or y alone. We will look at each type in turn.

$$\left(2y^3+2y^2\right)dx+\left(3y^2x+2xy\right)dy=0$$

Is it separable? Can we write it as $\dfrac{dy}{dx}=g(y)\cdot h(x)$?

Is it linear? Can we write it as $a_1(x)\dfrac{dy}{dx}+a_0(x)\cdot y=b(x)$?

We flip the traditional roles of x and y to see if *that* is linear.

What about writing it in the form $a_1(y)\dfrac{dx}{dy}+a_0(y)\cdot x=b(y)$?

expl 1 continued: Our equation is $\left(2y^3+2y^2\right)dx+\left(3y^2x+2xy\right)dy=0$.

Is it exact? For $Mdx+Ndy=0$, do we have $M_y = N_x$?

Does the diff. eq. have an integrating factor? Can we show that $f=\left(M_y-N_x\right)/N$ depends only on x or that $f=\left(N_x-M_y\right)/M$ depends only on y?

When solving, we need only one of these to be true. We would choose the μ that would result in an easier solution.

Sidebar: In general, if $Mdx + Ndy = 0$ is *not* exact but $\mu(x)Mdx + \mu(x)Ndy = 0$ is exact, then $\frac{d}{dy}\left(\mu(x)\cdot M\right) = \frac{d}{dx}\left(\mu(x)\cdot N\right)$. (That is the test for exactness.) You could solve this to find $\mu(x)$. Khan Academy does this for a specific diff. eq. You could, theoretically at least, assume μ to be a function in both x and y and do the same thing. That would look like $\frac{d}{dy}\left(\mu(x,y)\cdot M\right) = \frac{d}{dx}\left(\mu(x,y)\cdot N\right)$ (Or assume μ to be a function in just y and solve for it.) Solving for μ in these equations can be very complicated. The problems we are given will be of the simpler forms discussed in Theorem 3.

First, check it for exactness. It will *not* be exact. Then follow Theorem 3 to find the integrating factor.

expl 2: Solve the diff. eq.
$\left(3x^2 + y\right)dx + \left(x^2y - x\right)dy = 0$

Let's drop the absolute value from $\mu(x)$. We will see why we are allowed to do this on the next page.

Determine $\mu(x)$ and proceed to the next page.

Exploration of example 2: Justifying our use of $\mu(x)$:

We chose $\mu(x) = x^{-2}$ and *not* $\mu(x) = |x|^{-2}$ as truly came out in the calculation. Were we justified in doing so?

While you can justify it because $|x|^2 = x^2$, we will focus on reasoning that would work if we had ended up with an odd exponent, like $\mu(x) = |x|^{-3}$ (where we could *not* simply say $|x|^3 = x^3$).

Our goal is to find a function that, when multiplied in, will make the original equation exact. So, verify that the new equation (after multiplying by $\mu(x) = x^{-2}$) is exact. Here it is.

$$x^{-2}\left(3x^2 + y\right)dx + x^{-2}\left(x^2 y - x\right)dy = 0$$

Be sure this is exact. Then,
follow the four-step process for
solving exact equations. Write small.

Additional (Lost) Solutions: As we saw on page 1, the value that makes $\mu(x) = x^{-2}$ undefined may also be a solution. What value of x are we talking about? Does it make the original diff. eq. true? If so, add that value to the solution as you wrote it above.

expl 3: Find the integrating factor in the form $x^n y^m$ and solve the diff. eq.

$\left(12+5xy\right)dx+\left(6xy^{-1}+3x^2\right)dy=0$

First, check it for exactness. It will *not* be exact. Then multiply it by $x^n y^m$ and pull out M and N from the new equation.

For this new equation to be exact, we want $M_y = N_x$. How does that help us find n and m in the integrating factor $x^n y^m$?

Of course, once we find $x^n y^m$, we still need to solve the exact equation. There is more room on the next page.

(extra room for work)

Do *not* forget about lost or gained solutions. A solution is gained *only* if it is a solution to our amended equation but *not* the original.

Worksheet: Integrating Factors Worksheet:
This will give you a bit of practice determining if an equation is separable, linear, exact, or if it has an integrating factor that is a function of x or y alone. Also, there is a non-exact equation to solve by the method of this section.