

Differential Equations
Class Notes
Special Integrating Factors (Section 2.5)

If a diff. eq. is *not* exact, we will multiply it by a function $\mu(x, y)$ that will make it exact.

Theoretically, an integrating factor exists for *all* differential equations in the form $M(x, y)dx + N(x, y)dy = 0$. However, *no* general rule is known. Methods are known for certain types and tend to be somewhat messier than what we see here.

In this section, we look at a particular case. We will start with a non-exact diff. eq. and transform it into an exact equation. Then we can use the solution method from the previous section.

Definition: Integrating Factor: If the equation $Mdx + Ndy = 0$ is *not* exact but $\mu(x, y)Mdx + \mu(x, y)Ndy = 0$ is exact, then the function $\mu(x, y)$ is called an **integrating factor** of the original equation. Here, M and N are functions in both x and y , or rather $M = M(x, y)$ and $N = N(x, y)$.

We saw this in the last example in the Class Notes for Exact Equations.

Once we determine $\mu(x, y)$, or are given it, we multiply our non-exact equation by it and the equation becomes exact.

But wait! Since we multiply by $\mu(x, y)$, it is possible that we **lose or gain solutions**. We must always check whether any solutions to $\mu(x, y) = 0$ are *truly* solutions to the *original* diff. eq. In addition, we will also find any values that make $\mu(x, y)$ undefined as these may be solutions.

How do we find $\mu(x, y)$?

Lost (additional) solutions may exist where $\mu(x, y)$ is undefined. **Extraneous (gained) solutions** may exist when it's zero.

We have a theorem.

Theorem 3: Special Integrating Factors:

Consider the *non*-exact equation $M(x, y)dx + N(x, y)dy = 0$.

Case 1: If $f = (M_y - N_x)/N$ depends only on x , then $\mu(x) = e^{\int f dx}$ is an integrating factor for the diff. eq.

Case 2: If $f = (N_x - M_y)/M$ depends only on y , then $\mu(y) = e^{\int f dy}$ is an integrating factor for the diff. eq.

Here, I diverge from the book's notation, preferring to use the notation $M_y = \frac{\partial M}{\partial y}$ and $N_x = \frac{\partial N}{\partial x}$. I also use this construction f to simplify the write-up.

The book contends that f should be continuous but this is *not* mentioned in other resources.

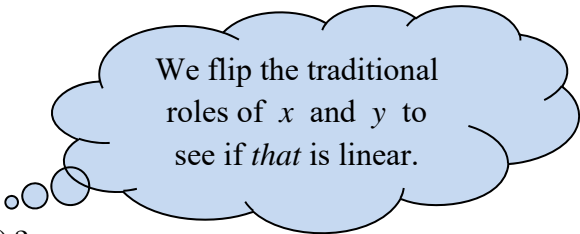
In order to solve the various diff. eqs. that we are given, we must be able to categorize them.

expl 1: Identify the diff. eqs. as separable, linear, exact, or having an integrating factor that is a function of either x or y alone. We will look at each type in turn.

$$(2y^3 + 2y^2)dx + (3y^2x + 2xy)dy = 0$$

Is it separable? Can we write it as $\frac{dy}{dx} = g(y) \cdot h(x)$?

Is it linear? Can we write it as $a_1(x)\frac{dy}{dx} + a_0(x) \cdot y = b(x)$?



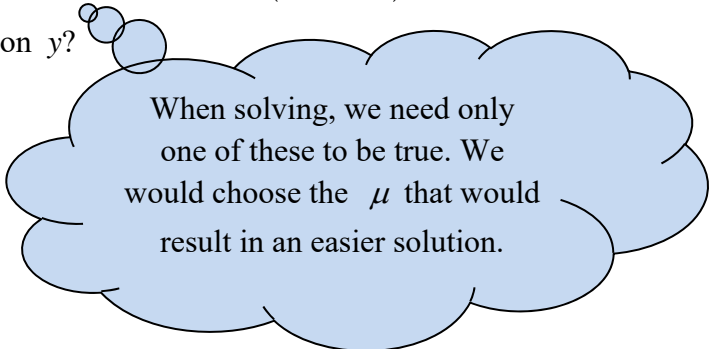
We flip the traditional roles of x and y to see if *that* is linear.

What about writing it in the form $a_1(y)\frac{dx}{dy} + a_0(y) \cdot x = b(y)$?

expl 1 continued: Our equation is $(2y^3 + 2y^2)dx + (3y^2x + 2xy)dy = 0$.

Is it exact? For $Mdx + Ndy = 0$, do we have $M_y = N_x$?

Does the diff. eq. have an integrating factor? Can we show that $f = (M_y - N_x)/N$ depends only on x or that $f = (N_x - M_y)/M$ depends only on y ?



When solving, we need only one of these to be true. We would choose the μ that would result in an easier solution.

Sidebar: In general, if $Mdx + Ndy = 0$ is *not* exact but $\mu(x)Mdx + \mu(x)Ndy = 0$ is exact,

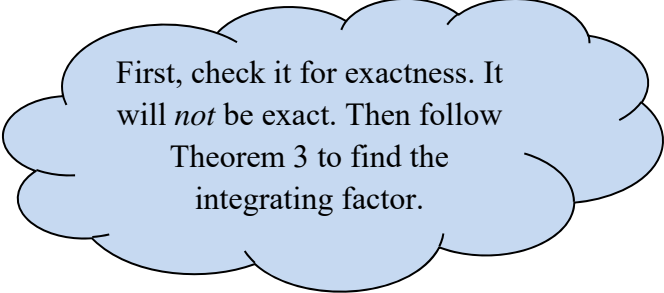
then $\frac{d}{dy}(\mu(x) \cdot M) = \frac{d}{dx}(\mu(x) \cdot N)$. (That is the test for exactness.) You could solve this to find

$\mu(x)$. Khan Academy does this for a specific diff. eq. You could, theoretically at least, assume μ to be a function in both x and y and do the same thing. That would look like

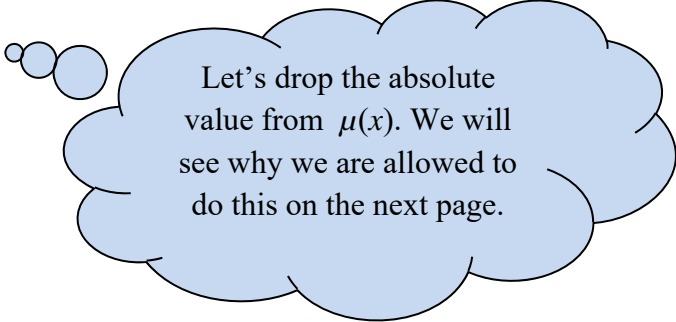
$$\frac{d}{dy}(\mu(x, y) \cdot M) = \frac{d}{dx}(\mu(x, y) \cdot N) \text{ (Or assume } \mu \text{ to be a function in just } y \text{ and solve for it.)}$$

Solving for μ in these equations can be very complicated. The problems we are given will be of the simpler forms discussed in Theorem 3.

expl 2: Solve the diff. eq. $(3x^2 + y)dx + (x^2y - x)dy = 0$



First, check it for exactness. It will *not* be exact. Then follow Theorem 3 to find the integrating factor.



Let's drop the absolute value from $\mu(x)$. We will see why we are allowed to do this on the next page.

Determine $\mu(x)$ and proceed to the next page.

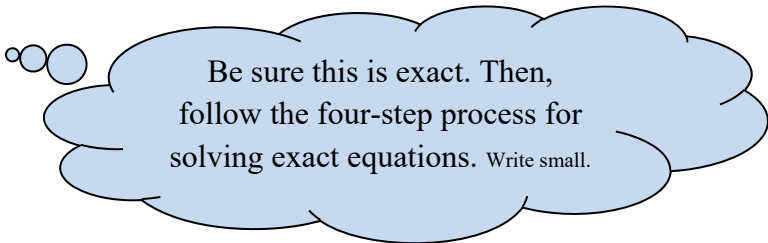
Exploration of example 2: Justifying our use of $\mu(x)$:

We chose $\mu(x) = x^{-2}$ and *not* $\mu(x) = |x|^{-2}$ as truly came out in the calculation. Were we justified in doing so?

While you can justify it because $|x|^2 = x^2$, we will focus on reasoning that would work if we had ended up with an odd exponent, like $\mu(x) = |x|^{-3}$ (where we could *not* simply say $|x|^3 = x^3$).

Our goal is to find a function that, when multiplied in, will make the original equation exact. So, verify that the new equation (after multiplying by $\mu(x) = x^{-2}$) is exact. Here it is.

$$x^{-2}(3x^2 + y)dx + x^{-2}(x^2y - x)dy = 0$$



Additional (Lost) Solutions: As we saw on page 1, the value that makes $\mu(x) = x^{-2}$ undefined may also be a solution. What value of x are we talking about? Does it make the original diff. eq. true? If so, add that value to the solution as you wrote it above.

expl 3: Find the integrating factor in the form $x^n y^m$ and solve the diff. eq.

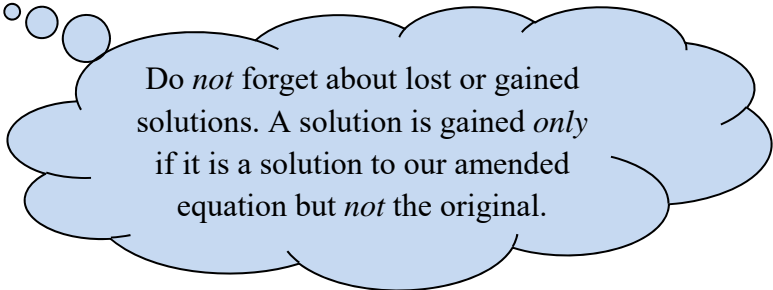
$$(12 + 5xy)dx + (6xy^{-1} + 3x^2)dy = 0$$

First, check it for exactness. It will *not* be exact. Then multiply it by $x^n y^m$ and pull out M and N from the new equation.

For this new equation to be exact, we want $M_y = N_x$. How does that help us find n and m in the integrating factor $x^n y^m$?

Of course, once we find $x^n y^m$, we still need to solve the exact equation. There is more room on the next page.

(extra room for work)



Do *not* forget about lost or gained solutions. A solution is gained *only* if it is a solution to our amended equation but *not* the original.

Worksheet: Integrating Factors Worksheet:

This will give you a bit of practice determining if an equation is separable, linear, exact, or if it has an integrating factor that is a function of x or y alone. Also, there is a non-exact equation to solve by the method of this section.