> Can we verify that a function is indeed a solution to a given diff. eq.?

Differential Equations
Class Notes
Solutions and Initial Value Problems (Section 1.2)

Our starting point and generic form:
The general form of an n^{th} order diff. eq. with x independent, y dependent, can be expressed as $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$. Here F is a function that depends only on x, y, and the derivatives of y up to order n. We assume this equation holds for all x in an open interval I ($a < x < b$ where a and b could be infinity).

In many cases, we isolate $\frac{d^ny}{dx^n}$ to get $\frac{d^ny}{dx^n} = f\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right)$.

> Notice how we use upper and lower cases for F and f.

Definition: Explicit solution: A function $\Phi(x)$, that when substituted for y into the diff. eq. satisfies the equation for all $x \in I$, is called an **explicit solution** to the diff. eq..

We will actually be solving differential equations later. For now we are merely verifying that a given function really is a solution to our diff. eq..

expl 1: Show that $\Phi(x) = e^x - x$ is an explicit solution to the following diff. eq. on the interval $(-\infty, \infty)$.

$$\frac{dy}{dx} + y^2 = e^{2x} + (1-2x)e^x + x^2 - 1$$

> We use this $\Phi(x)$ as y. What do we need to see if it makes the equation true?

Definition: Implicit solution: A relation $G(x, y) = 0$ is said to be an **implicit solution** of a diff. eq. on the interval I if it defines one or more explicit solutions on I.

expl 2: Show that $xy^3 - xy^3 \sin(x) = 1$ is an implicit solution to the following diff. eq. on the interval $\left(0, \pi/2\right)$.

$$\frac{dy}{dx} = \frac{\left(x\cos x + \sin x - 1\right) y}{3(x - x\sin x)}$$

First, solve the proposed solution for y or use implicit differentiation (if we know y is differentiable).

Does it make the diff. eq. true?

Remember that when we check to see if something is a solution, we *cannot* assume the equation is true. What does that mean?

Hey, what about that interval? … Huh? Oh yeah, right …

We were told to show that $xy^3 - xy^3\sin(x) = 1$ is an implicit solution to

$\dfrac{dy}{dx} = \dfrac{(x\cos x + \sin x - 1)\,y}{3(x - x\sin x)}$ *on the interval* $\left(0, \pi/2\right)$. What that amounts to, after we show that the function y *does* make the diff. eq. true, is that the x-values in this interval do *not* make either y or $\dfrac{dy}{dx}$ undefined. (In general, we do *not* want y or $\dfrac{d^n y}{dx^n} = f\left(x, y, \dfrac{dy}{dx}, \ldots, \dfrac{d^{n-1}y}{dx^{n-1}}\right)$ to be undefined in this interval.)

This is akin to showing that the solution $x = 11/4$ *not* only makes the equation $\dfrac{3}{x-2} = 4$ true but does *not* make any part undefined.

In practice, let's find where y and $\dfrac{dy}{dx}$ are undefined. Those x-values should be excluded from the intervals over which the solution makes the diff. eq. true.

Now, we have found $y = \left(x - x\sin x\right)^{-1/3}$ and $\dfrac{dy}{dx} = \dfrac{\sin x + x\cos x - 1}{3(x - x\sin x)^{4/3}}$.

These are both undefined when $\left(x - x\sin x\right) = 0$. Solve this to find the x-values for which y and $\dfrac{dy}{dx}$ are undefined.

Are any of these troublesome x-values in the interval $\left(0, \pi/2\right)$? What if we were asked about the interval $\left(0, \pi\right)$?

Definition: One-parameter family of solutions: A collection of all solutions of a diff. eq. which uses a constant such as $C \in \mathbb{R}$. (If there are two constants used, we call it a two-parameter family of solutions.)

expl 3: Verify that $x^2 + cy^2 = 1$ where $c \in \mathbb{R}$, $c \neq 0$, is a one-parameter family of implicit solutions to the following diff. eq.. Graph several solution curves on the same axes.

$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$

Let's use implicit differentiation, solving for $\frac{dy}{dx}$.

If $x^2 + cy^2 = 1$, then $-cy^2 = x^2 - 1$.

To graph, solve for y and choose values for c.

Definition: Initial value problem: We will find the solution to an n^{th} order diff. eq. $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$ on an interval I that satisfies, at x_0, the n initial conditions

$y(x_0) = y_0$, $\frac{dy}{dx}(x_0) = y_1$, ... $\frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}$. Here, $x_0 \in I$ and y_i are given constants.

Theorem 1: Existence and Uniqueness of Solution:

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. If f and $\frac{\partial f}{\partial y}$ are continuous functions in some rectangle $R = \{(x, y) : a < x < b; c < y < d\}$ that contains some point (x_0, y_0), then the initial value problem has a unique solution $\Phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$ where $\delta \in \mathbb{R}^+$.

Do you recognize delta?

expl 4: Determine whether Theorem 1 implies that the given initial value problem has a unique solution.

$$\frac{dy}{dt} - ty = \sin^2 t, \quad y(\pi) = 5$$

Rewrite diff. eq. in theorem's form to pick out $f(t, y)$. Then we find $\frac{\partial f}{\partial y}$. Are f and $\frac{\partial f}{\partial y}$ continuous around $(\pi, 5)$?

We will solve these initial value problems later.

Proving continuity in example 4: We said that the function $f(t, y) = ty + \sin^2 t$ was continuous around the point (π, 5). Let's prove it.

Recall: From calculus, we know a function $f(x, y)$ is **continuous** at (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$. A function is said to be continuous over an interval $[a, b]$ if it is continuous at each point in the interval. This implies that $f(x_0, y_0)$ and the limit exists.

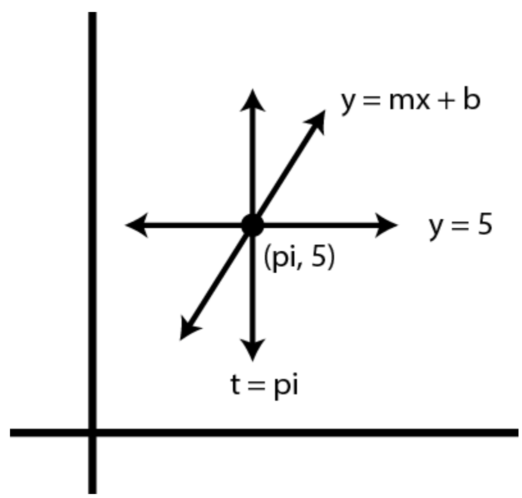
First, $f(\pi, 5) = 5\pi + \sin^2 \pi = 5\pi$.

Now we must determine if (with a change of variable) $\lim_{(t,y)\to(\pi,5)} f(t, y) = f(\pi, 5)$. If we show this is true, then we can say f is continuous.

So, we need to find $\lim_{(t,y)\to(\pi,5)} f(t, y) = \lim_{(t,y)\to(\pi,5)} ty + \sin^2 t$. We do this by looking at $f(t, y)$, approaching the point (π, 5) from all directions.

First, find $\lim_{(\pi,y)\to(\pi,5)} ty + \sin^2 t$. (This approaches the point (π, 5) along the vertical line $t = \pi$.)

Second, find $\lim_{(t,5)\to(\pi,5)} ty + \sin^2 t$. (This approaches the point (π, 5) along the horizontal line $y = 5$.)



This helps visualize us approaching the point (π, 5) along the vertical and horizontal lines. We will see how $y = mx + b$ plays a role next.

Notice these two limits agree.

Third, we will show this limit has the same value, no matter the slanted line on which we approach. Any slanted line ($y - y_1 = m(t - t_1)$) through the point $(\pi, 5)$ has the equation $y - 5 = m(t - \pi)$ or $y = mt - m\pi + 5$. So, we find the following limit.

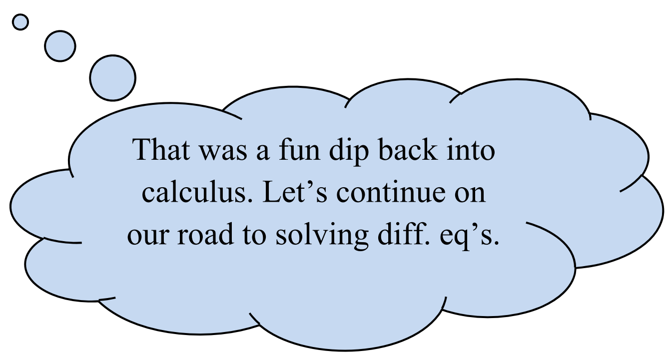
$$\lim_{\substack{t \to \pi \\ y = mt - m\pi + 5}} ty + \sin^2 t = \lim_{\substack{t \to \pi \\ y = mt - m\pi + 5}} t(mt - m\pi + 5) + \sin^2 t = \lim_{t \to \pi} \left(mt^2 - mt\pi + 5t + \sin^2 t \right)$$

As we let t approach π, we see this limit is also 5π. Did you show it in the space above?

Now, we can safely say that $\lim_{(t,y) \to (\pi,5)} f(t,y) = 5\pi$.

Therefore, we have shown that $\lim_{(t,y) \to (\pi,5)} f(t,y) = f(\pi,5)$ because both are 5π. Hence, the function f is continuous around $(\pi, 5)$.

Since $\partial f / \partial y$ was much simpler (in fact, $\partial f / \partial y = t$), we do *not* need so much work to show that it is continuous as well.



expl 5: Consider the initial value problem below. Explore it to show that the partial derivative of $f(x, y)$ with respect to y is *not* continuous at the initial point. (In fact, it will *not* be defined.) Therefore, Theorem 1 will *not* apply. Further, show that *both* $\Phi_1(x) = \frac{25}{4}(x-3)^2$ and $\Phi_2(x) = 0$ are solutions.

$$\frac{dy}{dx} = 5y^{1/2}, \quad y(3) = 0$$

When we *cannot* say Theorem 1 applies, we are saying that the solution, if it exists at all, is *not* unique.

expl 6: Determine for which values of m the function $\Phi(x) = x^m$ is a solution to the diff. eq. below.

$$3x^2\left(\frac{d^2y}{dx^2}\right)+11x\left(\frac{dy}{dx}\right)-3y=0$$

Let $\Phi(x) = x^m$ be the function y. Then find dy/dx and d^2y/dx^2.

After substitution, you will end up with a quadratic equation.