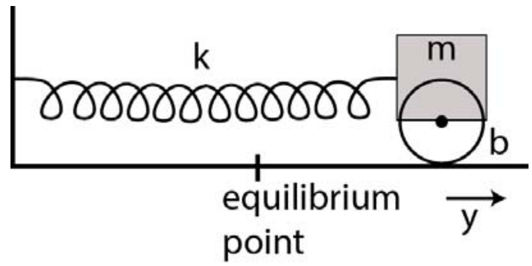
We will explore applications that result in linear, second-order diff. eqs.

Differential Equations
Class Notes
Introduction: The Mass-Spring Oscillator (Section 4.1)

Consider a spring attached to a wall with a mass on the other end. Below I have drawn this situation.

Newton's Second Law, $F = ma$, can be thought of as a second-order diff. eq. since $a = d^2y/dt^2$ where $y(t)$ is the **position** (or **motion**) **function** for the **mass**.



If the spring is unstretched and the inertial mass is still, then the system is at **equilibrium**. When the mass is moved (along the floor), the spring is compressed or stretched and it exerts a force on m that resists the displacement.

For most springs, this force is directly proportional to the displacement y and is thus given by $F_{spring} = -ky$ where $k > 0$ is known as the **stiffness** and the negative sign reflects the opposing nature of the force. (This is Hooke's Law and is only valid for sufficiently small displacements.)

We also have friction, $F_{friction} = -b\left(dy/dt\right) = -by'$ where $b \geq 0$ is the **damping coefficient**. It's negative, again, because it opposes the motion.

Other forces are considered external. For now, we will simply label these as F_{ext}.

So, now we have $F = ma = my'' = -ky - by' + F_{ext}$. Isolating the external force gives us a diff. eq. we can solve, $my'' + by' + ky = F_{ext}$. We will solve this for y to derive a function for the motion (position) of the mass.

Let's slide into this by first verifying a solution. We will later move on to solving these equations for ourselves.

expl 1: Verify that the exponentially damped sinusoid $y(t) = e^{-3t} \cdot \sin\left(\sqrt{3} \cdot t\right)$ is a solution to $my'' + by' + ky = F_{ext}$ if $F_{ext}(t) = 0$, $m = 1$, $b = 6$, and $k = 12$. What is the limit of this solution as $t \to \infty$?

A sinusoid is a curve that has the form of a sine wave.

First, put the values for m, b, and k in place to get our spring's diff. eq.

Find y' and y''. Does the proposed solution make our diff. eq. true?

Definition: Linear, Second-order Differential Equation: an equation of the form $ay'' + by' + cy = f(t)$ where $y(t)$ is an unknown function, $a,b,c \in \mathbb{R}$, and $f(t)$ is a known function.

Of course, we see this in the mass-spring application as $my'' + by' + ky = F_{ext}$.

Synchronous Solutions:

Now, if a mass-spring system is driven by an external force that is sinusoidal at the angular frequency ω (omega), then the system may be erratic at first but will eventually respond in "sync" with the driver and oscillate at the same frequency.

Examples of systems vibrating in synchronization with their drivers are sound system speakers, cyclists bicycling over railroad tracks, and ocean tides driven by the periodic pull of the moon.

Systems can be very sensitive to the particular frequency ω at which they are driven – like crystal shattering from a musical note or wind taking down a bridge.

Upper case omega: Ω
Lower case omega: ω

expl 2: Find a synchronous solution of the form $A\cos(\Omega t) + B\sin(\Omega t)$ to the given forced oscillator equation using the method of the book's example 4 to solve for A and B.
$y'' + 2y' + 5y = -50\sin(5t), \quad \Omega = 5$

Plan: Let $y = A\cos(\Omega t) + B\sin(\Omega t)$. Then find y' and y'', plugging them into the diff. eq. and solve for A and B with a system of equations.