What? Is that a quadratic equation? Are you serious?

Differential Equations
Class Notes
Homogeneous Linear Equations: The General Solution (Section 4.2)

When no external force acts on a mass-spring system, our equation $my'' + by' + ky = F_{ext}$ becomes $my'' + by' + ky = 0$. This happens when the spring vibrates freely.

In general, we saw in the previous section, $ay'' + by' + cy = f(t)$ is a linear, second-order diff. eq.. This $f(t)$ is called the "nonhomogeneity" in the equation.

So, set $f(t) = 0$ and we get $ay'' + by' + cy = 0$. This is called the **homogeneous form of the general linear, second-order diff. eq.**. (This is unrelated to the term "homogeneous" we saw in an earlier section where we solved equations by the same name.)

Notice, this y'' could be expressed in terms of y and y' because $y'' = \frac{1}{a}(-by' - cy)$.

A solution of the form $y = e^{rt}$ (where r is a constant) *could* fit since its derivatives are just constants times e^{rt}. Let's stick this "solution" into the equation $ay'' + by' + cy = 0$ (where $y' = re^{rt}$ and $y'' = r^2e^{rt}$). This would yield $ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ which simplifies to $e^{rt}(ar^2 + br + c) = 0$.

Now, e^{rt} is never 0 but solving $ar^2 + br + c = 0$ to find r (and therefore the solution $y = e^{rt}$) is a matter of solving a quadratic equation. In fact, we have a theorem.

Theorem:
The function $y = e^{rt}$ is a solution to $ay'' + by' + cy = 0$ *if and only if* r is a solution to $ar^2 + br + c = 0$ (or rather, $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$).

This equation $ar^2 + br + c = 0$ is called the **auxiliary** (or **characteristic**) **equation** associated with the homogeneous equation $ay'' + by' + cy = 0$.

Some of these equations will be factorable; others will need the quadratic formula.

Recall: Discriminants: For the equation $ar^2 + br + c = 0$, we know the number and nature of the solutions based solely on the discriminant, $b^2 - 4ac$.

Recall that when $b^2 - 4ac < 0$, the equation has two complex solutions. Further, when $b^2 - 4ac > 0$, the equation has two real solutions, and when $b^2 - 4ac = 0$, the equation has one real solution.

> Our book uses r_1 and r_2 to denote the real solutions of the auxiliary equation.

expl 1: Find a general solution to the given diff. eq.. Check your solutions.

$y'' + 5y' + 6y = 0$

> Form and solve the auxiliary equation. You should end up with two solutions in the form $y = e^{rt}$.

Distinct Real Roots and General Solutions:

We found two solutions to the diff. eq. in the previous example. We saw how $y = e^{-2t}$ and $y = e^{-3t}$ really do make the original diff. eq. true. In addition, it turns out, we can also say that $y = c_1 e^{-2t} + c_2 e^{-3t}$ is a solution. In fact, this solution does *not* depend on the values of the constants! Here is a theorem.

How would
you prove this?

Theorem: Distinct Real Roots:
If the auxiliary equation $ar^2 + br + c = 0$ has distinct real roots r_1 and r_2, then both $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are solutions to $ay'' + by' + cy = 0$. In addition, $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is a general solution (where c_1 and c_2 are constants).

What's more? We can find the values of these constants if we are given initial values $y(t_0) = Y_0$ and $y'(t_0) = Y_1$. Yeah, we got a theorem.

Theorem 2:
If $y_1(t)$ and $y_2(t)$ are any two solutions to the diff. eq. $ay'' + by' + cy = 0$ that are linearly independent on $(-\infty, \infty)$, then unique constants c_1 and c_2 can *always* be found so that $y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$ satisfies the initial value problem $ay'' + by' + cy = 0$, $y(t_0) = Y_0$, $y'(t_0) = Y_1$ on $(-\infty, \infty)$.

What does **linearly**
independent mean?

Definition: Linearly independent: The pair of functions $y_1(t)$ and $y_2(t)$ are **linearly independent** on the interval I *if and only if* neither of them is a constant multiple of the other on all of I.

We would say that y_1 and y_2 are **linearly dependent** on each other if one is a constant multiple of the other.

Notice, that if $r_1 \neq r_2$, then $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ will always be linearly independent.

One homework problem will explore
this concept with 3 functions.

Additional Solution: We should take note that $y=0$ is always a solution to the diff. eq. $ay''+by'+cy=0$. However, given initial values may rule it out as a solution for an initial value problem.

It may have occurred to you that we have only explored one of the three possibilities for the discriminant, when it is positive and we end up with two solutions for the auxiliary equation. We see a second possibility here and we will see the third in the next section.

Theorem: Repeated (Real) Root:

If the auxiliary equation $ar^2+br+c=0$ has a repeated root r, then both $y_1(t)=e^{rt}$ and $y_2(t)=t\cdot e^{rt}$ are solutions to $ay''+by'+cy=0$. In addition, $y(t)=c_1e^{rt}+c_2t\cdot e^{rt}$ is a general solution (where c_1 and c_2 are constants).

Form and solve the auxiliary equation. Follow the theorem above to write the general solution.

expl 2: Find a general solution to the given diff. eq..
$y''+6y'+9y=0$

expl 3: Solve the initial value problem.
$z'' - 2z' - 2z = 0, \quad z(0) = 0, \quad z'(0) = 3$

We will need the quadratic formula here.

A wacky system of equations will help us solve for c_1 and c_2.

Will this always work?

Theorem 1: Existence and Uniqueness: Homogeneous Case:
For any real numbers a ($a \neq 0$), b, c, t_0, Y_0, Y_1, there exists a unique solution to the initial value problem $ay'' + by' + cy = 0$, $y(t_0) = Y_0$, $y'(t_0) = Y_1$. The solution is valid for all t in $(-\infty, \infty)$.

More about Linear Independence:
Recall that two functions are linearly independent if neither is a constant multiple of the other.

expl 4: Use the definition of linearly independent functions to determine if the two functions below are linearly dependent on the interval (0, 1).

$y_1(t) = \tan^2 t - \sec^2 t$

$y_2(t) = 3$

How can the trig identity $\tan^2\theta + 1 = \sec^2\theta$ help us here?

expl 5: Use the definition of linearly independent functions to determine if the two functions below are linearly dependent on the interval $(0,\,1)$.

$y_1(t) = t \cdot e^{2t}$

$y_2(t) = e^{2t}$

We see that one function is the other times t, which is clearly *not* a constant. So, these are linearly independent functions. Let's prove it by contradiction.

Worksheet: Solving Homogeneous Linear Equations with Real Roots:

We explore the solution method, taking the extra step to check our solutions. We will also investigate a third-order diff. eq. in a generalization of the method.