

Our quadratic auxiliary equation
may have a negative discriminant.

Auxiliary Equations with Complex Roots (Section 4.3)

In the previous section, we put our mass-spring oscillators on hold to study only exponential solutions to the linear, second-order constant coefficient equations. Here, we see that these mass-spring systems can give rise to an auxiliary equation that has complex roots.

Rationale for Solutions:

For the linear, second-order diff. eq. $ay'' + by' + cy = 0$, we have its auxiliary equation $ar^2 + br + c = 0$. When $b^2 - 4ac < 0$, the equation has two complex roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} \cdot i.$$

We will call these roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, defining $\alpha = -b/2a$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$.

By these definitions, we see that α and β are real numbers.

From the previous section, the solutions to our diff. eq. are $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.

Using the values we have developed for our roots, these solutions are now in the form

$$y_1(t) = e^{(\alpha + i\beta)t} \text{ and } y_2(t) = e^{(\alpha - i\beta)t}.$$

We can use the Maclaurin series and Euler's formula (described in the book) to further rewrite our solutions. We now have $y_1(t) = e^{(\alpha + i\beta)t} = e^{\alpha t} (\cos(\beta t) + i \sin(\beta t))$ and

$$y_2(t) = e^{(\alpha - i\beta)t} = e^{\alpha t} (\cos(\beta t) - i \sin(\beta t)).$$

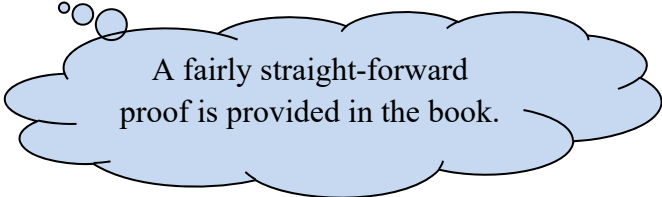
These solutions involve complex
numbers. Do they have to?

Finishing this out, we can say that $y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$ is a general solution to the diff. eq. $ay'' + by' + cy = 0$.

As the thought bubble at the bottom of the previous page ponders, we would like to find real-valued functions that are solutions to our diff. eq.. In fact, we have this lemma.

Lemma 2: Real Solutions Derived from Complex Solutions:

Let $z(t) = u(t) + i \cdot v(t)$ be a solution to $ay'' + by' + cy = 0$, where a, b , and c are real numbers. Then the real part (which is $u(t)$) and the imaginary part (which is $v(t)$) are real-valued solutions to the diff. eq..



This leads to our main theorem.

Theorem: Complex Conjugate Roots:

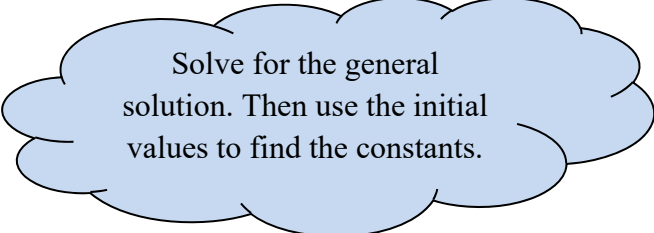
If the auxiliary equation has complex roots $\alpha \pm i\beta$, then two linearly independent solutions to $ay'' + by' + cy = 0$ are $y_1(t) = e^{\alpha t} \cos(\beta t)$ and $y_2(t) = e^{\alpha t} \sin(\beta t)$.

A general solution is given by $y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$ where c_1 and c_2 are real constants.

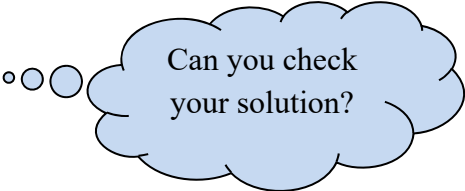
expl 1: The auxiliary equation for this diff. eq. has complex roots. Find a general solution.

$$y'' - 4y' + 7y = 0$$

expl 2: Solve the initial value problem.
 $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 1$



Solve for the general solution. Then use the initial values to find the constants.



Can you check your solution?

Vibrating Springs without Damping:

expl 3: A vibrating spring without damping can be modeled by the diff. eq. $my'' + by' + ky = 0$.

By taking $b = 0$ because there is no damping, this equation becomes $my'' + ky = 0$.

a.) If $m = 10$ kg, $k = 250$ kg/sec², $y(0) = 0.3$ m, and $y'(0) = -0.1$ m/sec, find the equation of motion for this undamped vibrating spring.

b.) After how many seconds will the mass in part *a* first cross the equilibrium point?

c.) When the equation of motion is of the form

$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$, the motion is said to be

oscillatory with frequency $\beta/2\pi$. Find the frequency of oscillation for the spring system in part *a*.

