> Our quadratic auxiliary equation may have a negative discriminant.

Differential Equations
Class Notes
Auxiliary Equations with Complex Roots (Section 4.3)

In the previous section, we put our mass-spring oscillators on hold to study only exponential solutions to the linear, second-order constant coefficient equations. Here, we see that these mass-spring systems can give rise to an auxiliary equation that has complex roots.

Rationale for Solutions:

For the linear, second-order diff. eq. $ay'' + by' + cy = 0$, we have its auxiliary equation $ar^2 + br + c = 0$. When $b^2 - 4ac < 0$, the equation has two complex roots

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} \cdot i .$$

We will call these roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, defining $\alpha = {}^{-b}\!/_{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$.
By these definitions, we see that α and β are real numbers.

From the previous section, the solutions to our diff. eq. are $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$.
Using the values we have developed for our roots, these solutions are now in the form
$y_1(t) = e^{(\alpha + i\beta)t}$ and $y_2(t) = e^{(\alpha - i\beta)t}$.

We can use the Maclaurin series and Euler's formula (described in the book) to further rewrite our solutions. We now have $y_1(t) = e^{(\alpha + i\beta)t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$ and
$y_2(t) = e^{(\alpha - i\beta)t} = e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)$.

> These solutions involve complex numbers. Do they have to?

Finishing this out, we can say that $y(t) = c_1 \cdot y_1(t) + c_2 \cdot y_2(t)$ is a general solution to the diff. eq. $ay'' + by' + cy = 0$.

As the thought bubble at the bottom of the previous page ponders, we would like to find real-valued functions that are solutions to our diff. eq.. In fact, we have this lemma.

Lemma 2: Real Solutions Derived from Complex Solutions:
Let $z(t)=u(t)+i\cdot v(t)$ be a solution to $ay''+by'+cy=0$, where a, b, and c are real numbers. Then the real part (which is $u(t)$) and the imaginary part (which is $v(t)$) are real-valued solutions to the diff. eq..

A fairly straight-forward proof is provided in the book.

This leads to our main theorem.

Theorem: Complex Conjugate Roots:
If the auxiliary equation has complex roots $\alpha \pm i\beta$, then two linearly independent solutions to $ay''+by'+cy=0$ are $y_1(t)=e^{\alpha t}\cos(\beta t)$ and $y_2(t)=e^{\alpha t}\sin(\beta t)$.

A general solution is given by $y(t)=c_1e^{\alpha t}\cos(\beta t)+c_2e^{\alpha t}\sin(\beta t)$ where c_1 and c_2 are real constants.

expl 1: The auxiliary equation for this diff. eq. has complex roots. Find a general solution.
$y''-4y'+7y=0$

expl 2: Solve the initial value problem.
$y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 1$

Solve for the general solution. Then use the initial values to find the constants.

Can you check your solution?

Vibrating Springs without Damping:

expl 3: A vibrating spring without damping can be modeled by the diff. eq. $my'' + by' + ky = 0$.
By taking $b = 0$ because there is no damping, this equation becomes $my'' + ky = 0$.
a.) If $m = 10$ kg, $k = 250$ kg/sec^2, $y(0) = 0.3$ m, and $y'(0) = -0.1$ m/sec, find the equation of motion for this undamped vibrating spring.
b.) After how many seconds will the mass in part *a* first cross the equilibrium point?
c.) When the equation of motion is of the form
$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$, the motion is said to be **oscillatory with frequency** $\beta/2\pi$. Find the frequency of oscillation for the spring system in part *a*.

Equilibrium means $y = 0$.