

Nonhomogeneous Equations: The Method of Undetermined Coefficients (Section 4.4)

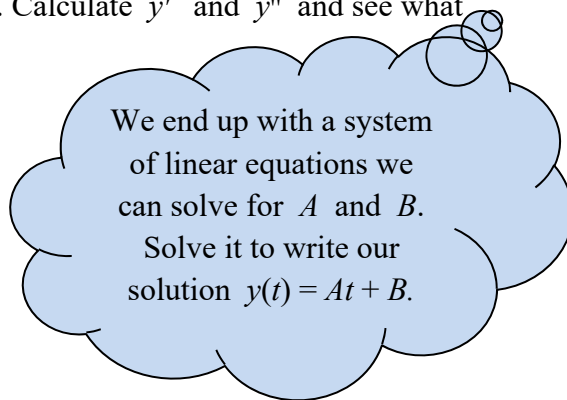
We have learned how to solve  $ay'' + by' + cy = f(t)$  where  $f(t) = 0$ . But what if this  $f(t)$  is *not* zero? In this section, this nonhomogeneity will be a single term of a certain type. It may seem crazy but we will *guess* at solutions to these equations. With some old-fashioned intuition, we can come up with a particular solution out of the infinite solutions that are out there.

**Rationale and Method for Nonhomogeneous Equations:**

Think about the equation  $y'' + 3y' + 2y = 3t$ . We must find a function  $y(t)$  such that  $y'' + 3y' + 2y$  is a linear function of  $t$  (in this case,  $3t$ ). What kind of function would fit? Perhaps a linear one?

Try  $y(t) = At$  (where  $A$  is a real number). We would calculate  $y' = A$  and  $y'' = 0$ . Put that all into the original equation and see if that works. Do you see a contradiction?

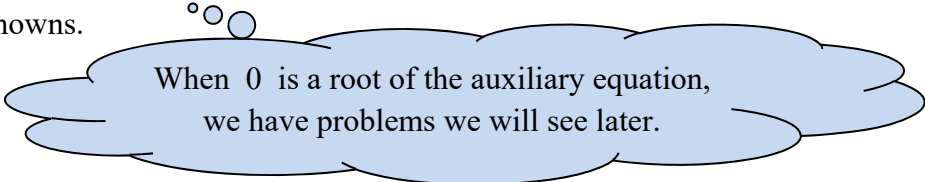
So, it does *not* work to simply use  $y(t) = At$ . Let's get a little more complicated in our guess. We'll try  $y(t) = At + B$  (where  $A$  and  $B$  are real numbers). Calculate  $y'$  and  $y''$  and see what you get when you put those into the original equation.



This is called the method of undetermined coefficients because we assume the solution to be of a certain type but with unknown (or, yet to be determined) coefficients.

### Method for a Certain Type of Nonhomogeneous Linear Equation:

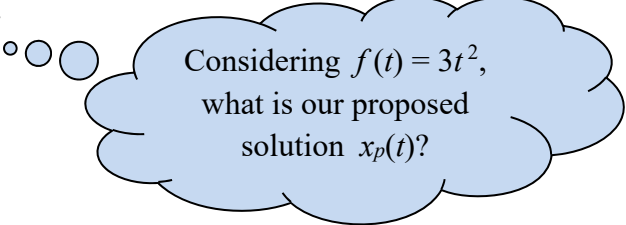
For the equation  $ay'' + by' + cy = Ct^m$  where  $m = 0, 1, 2, \dots$ , we guess a particular solution in the form  $y_p(t) = A_m t^m + \dots + A_1 t + A_0$ . These coefficients  $A_i$  are the **undetermined coefficients** we will find with a system of linear equations. In fact, we will solve a system of  $m + 1$  linear equations in  $m + 1$  unknowns.



When 0 is a root of the auxiliary equation, we have problems we will see later.

**Note:** We must retain all of the powers of  $t^m, t^{m-1}, \dots, t^0$  in the proposed solution even though they may *not* appear in the nonhomogeneity  $f(t)$ .

expl 1: Find a particular solution to the diff. eq..  
 $2x' + x = 3t^2$



Considering  $f(t) = 3t^2$ , what is our proposed solution  $x_p(t)$ ?

### **Different Types of Equations:**

That is all well and good but that only covers one type of equation we will encounter. What about  $y'' + 3y' + 2y = f(t)$  where  $f(t) = 10e^{3t}$  or  $f(t) = 2 \sin 5t$ ?

In the case of the above simple exponential nonhomogeneity, we will find that the function  $y(t) = Ae^{3t}$  will suffice. (Try it out!) When  $f(t)$  gets more complicated, we will need to get more creative.

In the case of  $f(t) = 2 \sin 5t$ , we need to remember that the derivative of sine is cosine. That implies the solution would include both sines and cosines, perhaps in the form  $y(t) = A \sin 5t + B \cos 5t$ .

### **Roots of Auxiliary Equations and Solutions to Nonhomogeneous Equations:**

The examples so far have worked out. (Phew!) But there are equations that will yield nonsensical solutions and the problems are rooted (ah, a pun!) in the related auxiliary equations. Let's explore.

expl 2: Consider the diff. eq.  $y'' + 3y' + 2y = 10e^{-2t}$  and its associated auxiliary equation  $r^2 + 3r + 2 = 0$ . First, find the roots of the auxiliary equation. Then, show that the proposed solution  $y(t) = Ae^{-2t}$  *cannot* be used to find a solution to the diff. eq.  $y'' + 3y' + 2y = 10e^{-2t}$ .

The problem that we are encountering here is dealt with a somewhat cumbersome theorem.

### Method of Undetermined Coefficients for Certain Single-Term Nonhomogeneities:

To find a particular solution to the diff. eq.  $ay'' + by' + cy = Ct^m e^{rt}$  where  $m$  is a non-negative integer, use the form  $y_p(t) = t^s (A_m t^m + \dots + A_1 t + A_0) e^{rt}$ . We use the following values for  $s$ .

- i.) Use  $s = 0$  if  $r$  is *not* a root of the associated auxiliary equation.
- ii.) Use  $s = 1$  if  $r$  is a *simple* root of the associated auxiliary equation.
- iii.) Use  $s = 2$  if  $r$  is a *double* root of the associated auxiliary equation.

To find a particular solution to the diff. eq.  $ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t} \cos \beta t \\ \text{OR} \\ Ct^m e^{\alpha t} \sin \beta t \end{cases}$  where  $\beta$  is

non-zero, use the form

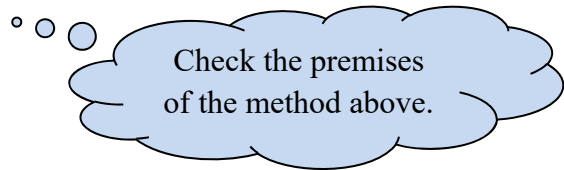
$$y_p(t) = t^s (A_m t^m + \dots + A_1 t + A_0) e^{\alpha t} \cos \beta t + t^s (B_m t^m + \dots + B_1 t + B_0) e^{\alpha t} \sin \beta t.$$

We use the following values for  $s$ .

- iv.) Use  $s = 0$  if  $\alpha + i\beta$  is *not* a root of the associated auxiliary equation.
- v.) Use  $s = 1$  if  $\alpha + i\beta$  is a *simple* root of the associated auxiliary equation.

expl 3: Decide if the method shown here can be used to solve the following equations. Explain.

a.)  $y'' + 2y' - y = t^{-1}e^t$

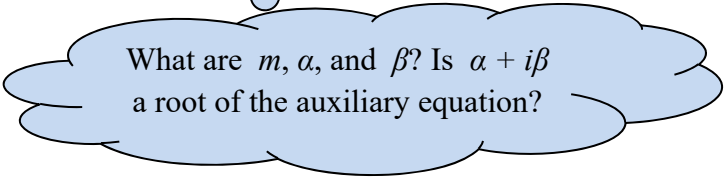


b.)  $2y''(x) - 6y'(x) + y(x) = \frac{\sin x}{e^{4x}}$

c.)  $y'' + 2y' - y = 4x \sin^2 x + 4x \cos^2 x$

expl 4: Find a particular solution to the diff. eq. below.

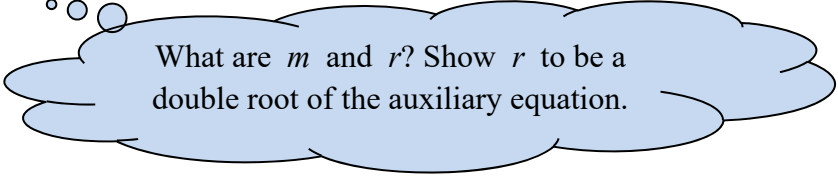
$$y'' - y' + 9y = 3 \sin(3t)$$



What are  $m$ ,  $\alpha$ , and  $\beta$ ? Is  $\alpha + i\beta$  a root of the auxiliary equation?

expl 5: Find a particular solution to the diff. eq. below.

$$x'' - 4x' + 4x = te^{2t}$$



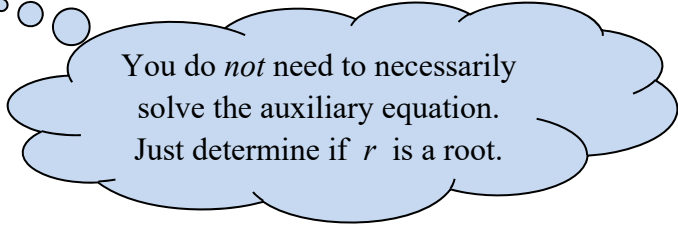
What are  $m$  and  $r$ ? Show  $r$  to be a double root of the auxiliary equation.

### Solving Higher-Order Linear Nonhomogeneous Equations:

We can extend our method for equations such as  $2y''' + 3y'' + y' - 4y = e^{-t}$  or  $y^{(4)} - 3y'' - 8y = \sin t$ . We simply have to determine the auxiliary equation (using  $r^3$  for  $y'''$ , etc.) and determine if  $r$  (from the term  $e^{rt}$ ) or  $\alpha + i\beta$  (from  $e^{\alpha t}\sin(\beta t)$ ) is a root.

expl 6: Find a particular solution to the higher-order diff. eq. below.

$$2y''' + 3y'' + y' - 4y = e^{-t}$$



You do *not* need to necessarily solve the auxiliary equation.  
Just determine if  $r$  is a root.