Recall the nonhomogeneity $f(t)$ from earlier and imagine it to be non-zero.

Differential Equations
Class Notes
Nonhomogeneous Equations: The Method of Undetermined Coefficients (Section 4.4)

We have learned how to solve $ay'' + by' + cy = f(t)$ where $f(t) = 0$. But what if this $f(t)$ is *not* zero? In this section, this nonhomogeneity will be a single term of a certain type. It may seem crazy but we will *guess* at solutions to these equations. With some old-fashioned intuition, we can come up with a particular solution out of the infinite solutions that are out there.

Rationale and Method for Nonhomogeneous Equations:

Think about the equation $y'' + 3y' + 2y = 3t$. We must find a function $y(t)$ such that $y'' + 3y' + 2y$ is a linear function of t (in this case, $3t$). What kind of function would fit? Perhaps a linear one?

Try $y(t) = At$ (where A is a real number). We would calculate $y' = A$ and $y'' = 0$. Put that all into the original equation and see if that works. Do you see a contradiction?

So, it does *not* work to simply use $y(t) = At$. Let's get a little more complicated in our guess. We'll try $y(t) = At + B$ (where A and B are real numbers). Calculate y' and y'' and see what you get when you put those into the original equation.

We end up with a system of linear equations we can solve for A and B. Solve it to write our solution $y(t) = At + B$.

This is called the method of undetermined coefficients because we assume the solution to be of a certain type but with unknown (or, yet to be determined) coefficients.

Method for a Certain Type of Nonhomogeneous Linear Equation:
For the equation $ay'' + by' + cy = Ct^m$ where $m = 0, 1, 2, \dots$, we guess a particular solution in the form $y_p(t) = A_m t^m + \dots + A_1 t + A_0$. These coefficients A_i are the **undetermined coefficients** we will find with a system of linear equations. In fact, we will solve a system of $m + 1$ linear equations in $m + 1$ unknowns.

When 0 is a root of the auxiliary equation,
we have problems we will see later.

Note: We must retain all of the powers of $t^m, t^{m-1}, \dots t^0$ in the proposed solution even though they may *not* appear in the nonhomogeneity $f(t)$.

expl 1: Find a particular solution to the diff. eq..
$2x' + x = 3t^2$

Considering $f(t) = 3t^2$,
what is our proposed
solution $x_p(t)$?

Different Types of Equations:
That is all well and good but that only covers one type of equation we will encounter. What about $y'' + 3y' + 2y = f(t)$ where $f(t) = 10e^{3t}$ or $f(t) = 2 \sin 5t$?

In the case of the above simple exponential nonhomogeneity, we will find that the function $y(t) = Ae^{3t}$ will suffice. (Try it out!) When $f(t)$ gets more complicated, we will need to get more creative.

In the case of $f(t) = 2 \sin 5t$, we need to remember that the derivative of sine is cosine. That implies the solution would include both sines and cosines, perhaps in the form
$y(t) = A \sin 5t + B \cos 5t$.

Roots of Auxiliary Equations and Solutions to Nonhomogeneous Equations:

The examples so far have worked out. (Phew!) But there are equations that will yield nonsensical solutions and the problems are rooted (ah, a pun!) in the related auxiliary equations. Let's explore.

expl 2: Consider the diff. eq. $y'' + 3y' + 2y = 10e^{-2t}$ and its associated auxiliary equation $r^2 + 3r + 2 = 0$. First, find the roots of the auxiliary equation. Then, show that the proposed solution $y(t) = Ae^{-2t}$ *cannot* be used to find a solution to the diff. eq. $y'' + 3y' + 2y = 10e^{-2t}$.

The problem that we are encountering here is dealt with a somewhat cumbersome theorem.

Method of Undetermined Coefficients for Certain Single-Term Nonhomogeneities:

To find a particular solution to the diff. eq. $ay'' + by' + cy = Ct^m e^{rt}$ where m is a non-negative integer, use the form $y_p(t) = t^s \left(A_m t^m + \ldots + A_1 t + A_0 \right) e^{rt}$. We use the following values for s.

i.) Use $s = 0$ if r is *not* a root of the associated auxiliary equation.
ii.) Use $s = 1$ if r is a *simple* root of the associated auxiliary equation.
iii.) Use $s = 2$ if r is a *double* root of the associated auxiliary equation.

To find a particular solution to the diff. eq. $ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t} \cos \beta t \\ \quad OR \\ Ct^m e^{\alpha t} \sin \beta t \end{cases}$ where β is non-zero, use the form

$$y_p(t) = t^s \left(A_m t^m + \ldots + A_1 t + A_0 \right) e^{\alpha t} \cos \beta t + t^s \left(B_m t^m + \ldots + B_1 t + B_0 \right) e^{\alpha t} \sin \beta t \,.$$

We use the following values for s.

iv.) Use $s = 0$ if $\alpha + i\beta$ is *not* a root of the associated auxiliary equation.
v.) Use $s = 1$ if $\alpha + i\beta$ is a *simple* root of the associated auxiliary equation.

expl 3: Decide if the method shown here can be used to solve the following equations. Explain.

a.) $y'' + 2y' - y = t^{-1} e^t$

Check the premises of the method above.

b.) $2y''(x) - 6y'(x) + y(x) = \dfrac{\sin x}{e^{4x}}$

c.) $y'' + 2y' - y = 4x \sin^2 x + 4x \cos^2 x$

expl 4: Find a particular solution to the diff. eq. below.

$y'' - y' + 9y = 3\sin(3t)$

What are m, α, and β? Is $\alpha + i\beta$ a root of the auxiliary equation?

expl 5: Find a particular solution to the diff. eq. below.

$x'' - 4x' + 4x = te^{2t}$

What are m and r? Show r to be a double root of the auxiliary equation.

Solving Higher-Order Linear Nonhomogeneous Equations:

We can extend our method for equations such as $2y''' + 3y'' + y' - 4y = e^{-t}$ or $y^{(4)} - 3y'' - 8y = \sin t$. We simply have to determine the auxiliary equation (using r^3 for y''', etc.) and determine if r (from the term e^{rt}) or $\alpha + i\beta$ (from $e^{\alpha t}\sin(\beta t)$) is a root.

expl 6: Find a particular solution to the higher-order diff. eq. below.
$2y''' + 3y'' + y' - 4y = e^{-t}$

You do *not* need to necessarily solve the auxiliary equation. Just determine if r is a root.