We will solve a linear first-order diff. eq. with a special method that forces it into a form with which we can work.

Differential Equations
Class Notes
Linear Equations (Section 2.3)

**Recall: Definition:** A **linear first-order differential equation** is of the form $\mathrm{a}_1(x)\dfrac{dy}{dx}+\mathrm{a}_0(x)\cdot y=b(x)$. Here, $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on $x$, not $y$. The **standard form** of a linear diff. eq. is $\dfrac{dy}{dx}+P(x)\cdot y=Q(x)$.

So, how do we solve them? We have two cases.

**Methods for Solving Linear First-order diff. eq.:**

**Case 1:** If $a_0(x) = 0$, then $\mathrm{a}_1(x)\dfrac{dy}{dx}=b(x)$ and you can solve by solving for $dy/dx$ and integrating. This would get us $\dfrac{dy}{dx}=\dfrac{b(x)}{\mathrm{a}_1(x)}$ and $y=\displaystyle\int\frac{b(x)}{\mathrm{a}_1(x)}+c$ for some $c\in\mathbb{R}$. This assumes that $a_1(x)$ is *not* equal to zero.

This case is rare.

**Case 2:** If $a_0(x)=a_1'(x)$, then the diff. eq. $\mathrm{a}_1(x)\dfrac{dy}{dx}+\mathrm{a}_0(x)\cdot y=b(x)$ becomes $\mathrm{a}_1(x)\dfrac{dy}{dx}+\mathrm{a}_1'(x)\cdot y=b(x)$. But do you recognize the left side?

This could be written $\dfrac{\mathrm{d}}{\mathrm{dx}}\left(\mathrm{a}_1(x)\cdot y\right)=b(x)$. We can integrate this to solve for $y$, getting

$$y=\frac{\int b(x)dx+c}{\mathrm{a}_1(x)}.$$

Case 2 seems like it would be rare too. But, it turns out that *any* linear 1$^{st}$-order diff. eq. can be turned into a case 2 equation by multiplying be an "integrating factor". We will call this factor $\mu(x)$.

The symbol $\mu$ is pronounced "mew".

What we will essentially be doing is multiplying our whole equation by $\mu(x)$. If we choose this $\mu(x)$ correctly, that will turn our equation into the form we saw back in case 2. The book justifies why we use the $\mu(x)$ as defined below.

**Method for Solving (Case 2) Linear First-order diff. eq.:**

a.) Write the equation in the standard form $\frac{dy}{dx} + P(x) \cdot y = Q(x)$.

b.) Calculate $\mu(x) = e^{\int P(x)dx}$.

The constant of integration can be anything, so choose zero.

c.) Multiply the equation by $\mu(x)$.

This yields $\mu(x) \cdot \frac{dy}{dx} + \mu(x) \cdot P(x) \cdot y = \mu(x) \cdot Q(x)$. More importantly, we see this is equal to $\frac{d}{dx}(\mu(x) \cdot y) = \mu(x) \cdot Q(x)$.

The book shows how $\mu'(x) = \mu(x) \cdot P(x)$.

Focus on this last form.

In practice, use this last form here to rewrite the equation.

d.) Integrate both sides and divide by $\mu(x)$ to solve for $y$. This gets us $y = \frac{\int \mu(x)Q(x)dx + c}{\mu(x)}$.

When we calculate $\mu(x) = e^{\int P(x)dx}$, we can take the *easiest to use* function that satisfies $\mu'(x) = \mu(x) \cdot P(x)$.

This $c$ comes from the integration.

**Definition: General solution:** This is the solution with the constant of integration above in place. We recently called this a **one-parameter family of solutions**.

expl 1: Obtain a general solution to the equation below.

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1$$

Follow steps *a* through *d*. Label them as you go.

Identify $P(x)$ and $Q(x)$. Calculate $\mu(x)$.

You will get $\mu(x) = \frac{1}{|x|}$.

Can we use $\mu(x) = \frac{1}{x}$?

Do *not* forget your constant of integration and solve for $y$.

**Initial Value Problems:**
We combine this solution method with an initial value problem set-up and get the following theorem.

**Theorem 1: Existence and Uniqueness of Solution:**
If $P(x)$ and $Q(x)$ are continuous on an interval $(a, b)$ that contains the point $x_0$, then for any choice of initial value $y_0$, there exists a unique solution $y(x)$ on $(a, b)$ to the initial value problem $\frac{dy}{dx} + P(x) \cdot y = Q(x), \quad y(x_0) = y_0$.

In fact, the solution is given by $y = \frac{\int \mu(x)Q(x)dx + c}{\mu(x)}$ for a suitable value of $c$.

You might need
$u$-substitution
or other tricks.

expl 2: Solve the initial value problem.
$\frac{dy}{dx} + 4y - e^{-x} = 0, \quad y(0) = 4/3$

Use the
initial point
to find $c$.

expl 3: **Application: Secretion of Hormones:** The secretion of hormones into the blood is often a periodic activity. If a hormone is secreted on a 24-hour cycle, then the rate of change in the level of the hormone in the blood may be represented by the initial value problem $\frac{dx}{dt} = \alpha - \beta\cos\left(\pi \cdot t/12\right) - kx, \quad x(0) = x_0$. Here, $x(t)$ is the level of the hormone in the blood at time $t$, $\alpha$ is the average secretion rate, $\beta$ is the amount of daily variation in the secretion, and $k$ is a positive constant reflecting the rate at which the body removes the hormone from the blood. If $\alpha = \beta = 1$, $k = 2$, and $x_0 = 10$, find $x(t)$.

Put it all in and try to write the diff. eq. in standard form.

Wacky! Integral formula #107

$$\int e^{bx}\cos(ax)\,dx = \frac{1}{a^2+b^2}\cdot e^{bx}\left(a\sin(ax) + b\cos(ax)\right)$$

(extra room for work)

**Is this equation linear?**
Sometimes an equation will *not appear* linear because we are thinking of the traditional roles of independent and dependent variables. We will see differential equations where, if we take the $x$ to be the independent and $y$ to be the dependent variables, it will *not* be linear. However, if we switch that and let the $y$ be the independent variable, it can be shown to be linear. This does *not* happen in this section but will in the next.

An example is $\theta dr + \left(3r - \theta - 1\right) d\theta = 0$. We will explore this later. We will see that, if we take $\theta$ as the dependent variable, it is *not* linear. However, if we take $r$ as the dependent variable, it can be shown to be linear.

**Worksheet: Separable and Linear Differential Equations Practice:**
This worksheet will give you a couple of diff. eq. to practice.