

We will solve a linear first-order diff. eq. with a special method that forces it into a form with which we can work.

**Recall: Definition:** A **linear first-order differential equation** is of the form

$a_1(x) \frac{dy}{dx} + a_0(x) \cdot y = b(x)$ . Here,  $a_1(x)$ ,  $a_0(x)$ , and  $b(x)$  depend only on  $x$ , not  $y$ . The **standard form** of a linear diff. eq. is  $\frac{dy}{dx} + P(x) \cdot y = Q(x)$ .

So, how do we solve them? We have two cases.

**Methods for Solving Linear First-order diff. eq.:**

**Case 1:** If  $a_0(x) = 0$ , then  $a_1(x) \frac{dy}{dx} = b(x)$  and you can solve by solving for  $dy/dx$  and

integrating. This would get us  $\frac{dy}{dx} = \frac{b(x)}{a_1(x)}$  and  $y = \int \frac{b(x)}{a_1(x)} + c$  for some  $c \in \mathbb{R}$ . This assumes

that  $a_1(x)$  is *not* equal to zero.

This case is rare.

**Case 2:** If  $a_0(x) = a_1'(x)$ , then the diff. eq.  $a_1(x) \frac{dy}{dx} + a_0(x) \cdot y = b(x)$  becomes

$a_1(x) \frac{dy}{dx} + a_1'(x) \cdot y = b(x)$ . But do you recognize the left side?

This could be written  $\frac{d}{dx}(a_1(x) \cdot y) = b(x)$ . We can integrate this to solve for  $y$ , getting

$$y = \frac{\int b(x) dx + c}{a_1(x)}.$$

Case 2 seems like it would be rare too. But, it turns out that *any* linear 1<sup>st</sup>-order diff. eq. can be turned into a case 2 equation by multiplying by an “integrating factor”. We will call this factor  $\mu(x)$ .

The symbol  $\mu$  is pronounced “mew”.

What we will essentially be doing is multiplying our whole equation by  $\mu(x)$ . If we choose this  $\mu(x)$  correctly, that will turn our equation into the form we saw back in case 2. The book justifies why we use the  $\mu(x)$  as defined below.

**Method for Solving (Case 2) Linear First-order diff. eq.:**

a.) Write the equation in the standard form  $\frac{dy}{dx} + P(x) \cdot y = Q(x)$ .

b.) Calculate  $\mu(x) = e^{\int P(x)dx}$ .

The constant of integration can be anything, so choose zero.

c.) Multiply the equation by  $\mu(x)$ .

This yields  $\mu(x) \cdot \frac{dy}{dx} + \mu(x) \cdot P(x) \cdot y = \mu(x) \cdot Q(x)$ . More importantly, we see this is equal to

$$\frac{d}{dx}(\mu(x) \cdot y) = \mu(x) \cdot Q(x).$$

The book shows how  $\mu'(x) = \mu(x) \cdot P(x)$ .

Focus on this last form.

In practice, use this last form here to rewrite the equation.

d.) Integrate both sides and divide by  $\mu(x)$  to solve for  $y$ . This gets us  $y = \frac{\int \mu(x)Q(x)dx + c}{\mu(x)}$ .

When we calculate  $\mu(x) = e^{\int P(x)dx}$ , we can take the *easiest to use* function that satisfies  $\mu'(x) = \mu(x) \cdot P(x)$ .

This  $c$  comes from the integration.

**Definition: General solution:** This is the solution with the constant of integration above in place. We recently called this a **one-parameter family of solutions**.

expl 1: Obtain a general solution to the equation below.

$$\frac{dy}{dx} = \frac{y}{x} + 2x + 1$$

Follow steps *a*  
through *d*. Label  
them as you go.

Identify  $P(x)$   
and  $Q(x)$ .  
Calculate  $\mu(x)$ .

You will get  $\mu(x) = \frac{1}{|x|}$ .

Can we use  $\mu(x) = \frac{1}{x}$ ?

Do *not* forget  
your constant  
of integration  
and solve for  $y$ .

### Initial Value Problems:

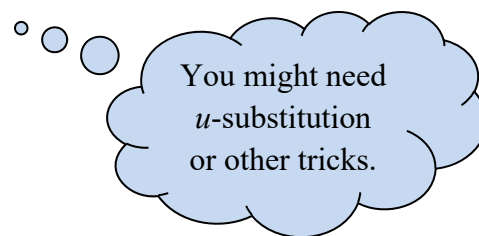
We combine this solution method with an initial value problem set-up and get the following theorem.

#### Theorem 1: Existence and Uniqueness of Solution:

If  $P(x)$  and  $Q(x)$  are continuous on an interval  $(a, b)$  that contains the point  $x_0$ , then for any choice of initial value  $y_0$ , there exists a unique solution  $y(x)$  on  $(a, b)$  to the initial value

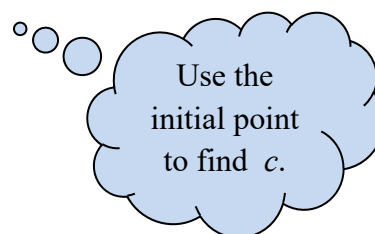
problem  $\frac{dy}{dx} + P(x) \cdot y = Q(x)$ ,  $y(x_0) = y_0$ .

In fact, the solution is given by  $y = \frac{\int \mu(x)Q(x)dx + c}{\mu(x)}$  for a suitable value of  $c$ .



expl 2: Solve the initial value problem.

$$\frac{dy}{dx} + 4y - e^{-x} = 0, \quad y(0) = \frac{4}{3}$$



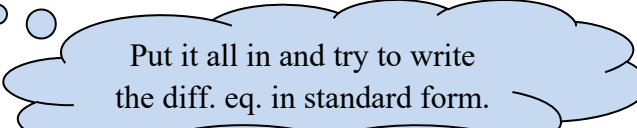
expl 3: **Application: Secretion of Hormones:** The secretion of hormones into the blood is often a periodic activity. If a hormone is secreted on a 24-hour cycle, then the rate of change in the level of the hormone in the blood may be represented by the initial value problem

$$\frac{dx}{dt} = \alpha - \beta \cos\left(\pi \cdot \frac{t}{12}\right) - kx, \quad x(0) = x_0.$$

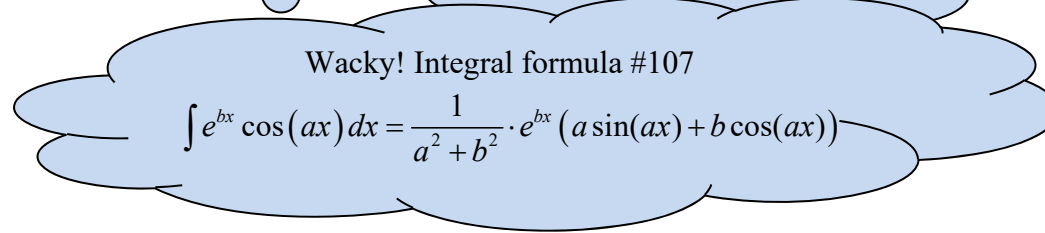
Here,  $x(t)$  is the level of the hormone in the blood at

time  $t$ ,  $\alpha$  is the average secretion rate,  $\beta$  is the amount of daily variation in the secretion, and  $k$  is a positive constant reflecting the rate at which the body removes the hormone from the blood.

If  $\alpha = \beta = 1$ ,  $k = 2$ , and  $x_0 = 10$ , find  $x(t)$ .



Put it all in and try to write the diff. eq. in standard form.



Wacky! Integral formula #107

$$\int e^{bx} \cos(ax) dx = \frac{1}{a^2 + b^2} \cdot e^{bx} (a \sin(ax) + b \cos(ax))$$

(extra room for work)

### **Is this equation linear?**

Sometimes an equation will *not appear* linear because we are thinking of the traditional roles of independent and dependent variables. We will see differential equations where, if we take the  $x$  to be the independent and  $y$  to be the dependent variables, it will *not* be linear. However, if we switch that and let the  $y$  be the independent variable, it can be shown to be linear. This does *not* happen in this section but will in the next.

An example is  $\theta dr + (3r - \theta - 1)d\theta = 0$ . We will explore this later. We will see that, if we take  $\theta$  as the dependent variable, it is *not* linear. However, if we take  $r$  as the dependent variable, it can be shown to be linear.

### **Worksheet: Separable and Linear Differential Equations Practice:**

This worksheet will give you a couple of diff. eq. to practice.