These problems will result in a *system* of differential equations.

Differential Equations
Class Notes
Introduction to Systems: Interconnected Fluid Tanks (Section 5.1)

Think back to our problems involving a single closed tank with brine circulating within. We will now add a second tank that is interconnected to the first. Here is a picture.

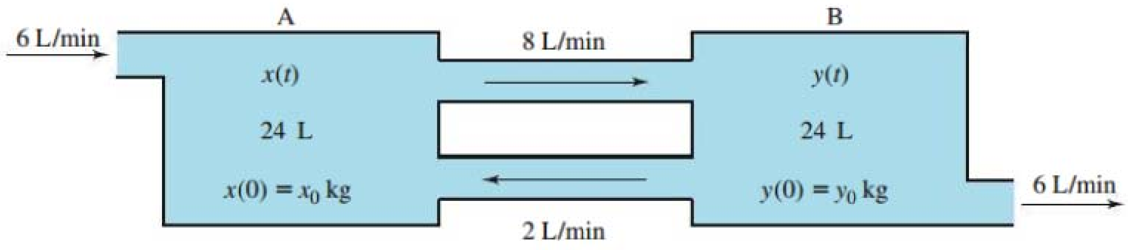

Figure 5.1 Interconnected fluid tanks

What we see here are two tanks, each with volume 24 liters (L). Tank A has $x(t)$ kilograms (kg) of salt dissolved in the water at time t minutes; Tank B has $y(t)$ kilograms (kg) of salt dissolved in the water at time t minutes. We will take t to be greater than zero. Fresh water enters tank A at the rate of 6 liters per minute on the left; briny water leaves tank B at the rate of 6 liters per minute on the right.

The tanks are interconnected. Briny water flows from tank A to tank B at a rate of 8 liters per minute; Briny water flows from tank B to tank A at a rate of 2 liters per minute. The liquids are kept well stirred so we assume they are homogeneous.

The initial *amounts* (kg) of salt in tanks A and B are, respectively, $x(0) = x_0$ and $y(0) = y_0$.

Since the system is being flushed with fresh water, it is true that as $t \to \infty$, both x and y diminish to zero.

Let's examine the flows and concentrations of salt more closely. As we have seen in a previous section, the salt *concentration* (kg/L) in tank A is $\frac{x(t)\ kg}{24\ L}$. The upper pipe carries salt *out* of tank A at a rate of $\frac{8\ L}{1\text{ min}} \cdot \frac{x\ kg}{24\ L} = \frac{x\ kg}{3\text{ min}}$. Similarly, the salt *concentration* (kg/L) in tank B is $\frac{y(t)\ kg}{24\ L}$. The lower pipe carries salt *into* tank A at a rate of $\frac{2\ L}{1\text{ min}} \cdot \frac{y\ kg}{24\ L} = \frac{y\ kg}{12\text{ min}}$.

This gives us $\frac{dx}{dt} = input\ rate - output\ rate = \frac{y}{12} - \frac{x}{3}$. Similarly, for tank B, we have

$$\frac{dy}{dt} = input\ rate - output\ rate$$
$$= \frac{x}{3} - \left(\frac{2y}{24} + \frac{6y}{24}\right)$$
$$= \frac{x}{3} - \frac{y}{3}$$

> Tank B has two outlet pipes.

Look at that! We have a system of equations. $\begin{aligned} x' &= -\frac{1}{3}x + \frac{1}{12}y \\ y' &= \ \ \frac{1}{3}x - \frac{1}{3}y \end{aligned}$.

Solve the second equation for x and substitute it into the first equation. This yields

$$x' = -\frac{1}{3}x + \frac{1}{12}y$$
$$(3y' + y)' = -\frac{1}{3}(3y' + y) + \frac{1}{12}y$$
$$3y'' + y' = -y' - \frac{1}{4}y$$
$$3y'' + 2y' + \frac{1}{4}y = 0$$

> This is a second-order, linear diff. eq. with constant coefficients.

As we have seen before, this can be solved by solving its auxiliary equation, $3r^2 + 2r + \frac{1}{4} = 0$.

We get $r = -\frac{1}{2}, -\frac{1}{6}$. That gives us a general solution of $y(t) = c_1 e^{-t/2} + c_2 e^{-t/6}$ where c_1 and c_2 are real numbers. Complete the process to find c_1 and c_2.

> Here, find y' and then x given that $x = 3y' + y$.

To finish this out, solve for c_1 and c_2 by recalling $x(0) = x_0$ and $y(0) = y_0$. This yields a system of equations involving c_1 and c_2. This process will be generalized in the next section to find solutions of all linear systems with constant coefficients.