We will study four types of equations and how to solve them.

Differential Equations
Class Notes
Substitutions and Transformations (Section 2.6)

When $M \cdot dx + N \cdot dy = 0$ is *not* separable, linear, or exact, we *may* still be able to solve it with a transformation (as seen, in part, in the previous section) or a substitution. Here, we see four types of equations that can be transformed into a separable or linear equation.

Type 1: Homogeneous Equations:

Definition: Homogeneous equation: If the right-hand side of the equation $\frac{dy}{dx} = f(x, y)$ can be expressed as a function of the ratio y/x alone, then the equation is **homogeneous**.

expl 1: Show this differential equation is homogeneous.
$2tx\,dx + (t^2 - x^2)dt = 0$

Solve for $\frac{dx}{dt}$ (or possibly $\frac{dt}{dx}$).

Test for Homogeneity: Replace x by tx and y by ty. Then $\frac{dy}{dx} = f(x,y)$ is homogeneous if and only if $f(tx,ty) = f(x,y)$ for all $t \neq 0$.

Rationale and Method for Solving Homogeneous Equations:

We'll use a substitution. Let $v = y/x$. Now, our equation would be of the form $\frac{dy}{dx} = G(v)$ for some function G. We need to express $\frac{dy}{dx}$ in terms of x and v.

Look at definition.

Since $v = y/x$, we have $y = x \cdot v$ and so, by the product rule, we know $\frac{dy}{dx} = \left(\frac{d}{dx}x\right) \cdot v + x \cdot \left(\frac{d}{dx}v\right) = 1 \cdot v + x \cdot \frac{dv}{dx}$. Substituting that for $\frac{dy}{dx}$ into the diff. eq. yields the following.

$$\frac{dy}{dx} = G(v)$$

$$v + x\frac{dv}{dx} = G(v)$$

$$x\frac{dv}{dx} = G(v) - v$$

This is separable!

$$\left(G(v) - v\right)^{-1} dv = \frac{1}{x}dx$$

Since this is separable, we solve by integrating both sides, $\int \frac{1}{G(v)-v} dv = \int \frac{1}{x} dx$.

Once solved, remember your solution will be in terms of v and x. So, our last step will be to resubstitute the original variables with $v = y/x$.

expl 2: Use the method of homogeneous equations to solve.

$$\left(3x^2 - y^2\right)dx + \left(xy - x^3 y^{-1}\right)dy = 0$$

Perform the test for homogeneity first. You do *not* need to solve for $\frac{dy}{dx}$ first.

To solve, rewrite the diff. eq. as $\frac{dy}{dx} = G(v)$ where $v = y/x$.

Simplify $G(v) - v$ as much as you can.

Do *not* forget that the solution contains x and y, *not* v.

(extra room for work)

Type 2: Equations of the Form $\frac{dy}{dx} = G(ax+by)$:

When the right-hand side of $\frac{dy}{dx} = f(x, y)$ can be expressed as a function of $(ax + by)$ where $a, b \in \mathbb{R}$, then we see it as $\frac{dy}{dx} = G(ax+by)$ and so the substitution $z = ax + by$ transforms the equation into a separable one.

expl 3a: Use the method for Equations of the Form $\frac{dy}{dx} = G(ax+by)$ to solve.

$$\frac{dy}{dx} = (x+y+2)^2$$

We will let $z = x + y$.

Our equation is now $\frac{dy}{dx} = (z+2)^2$.
Find ${dz}/{dx}$ in terms of z. This will be a separable equation.

Can you solve the final solution for y?

expl 3b: Let’s check the solution. Put the solution into the original equation and see if it makes it true.

Trig identity:
$\tan^2\theta + 1 = \sec^2\theta$

Type 3: Bernoulli Equations:

Definition: Bernoulli equation: A first-order diff. eq. that could be written in the form $\dfrac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n$ where $P(x)$ and $Q(x)$ are continuous on an interval (a, b) and $n \in \mathbb{R}$, is called a **Bernoulli equation**.

Does this form look familiar? What kind of equation is this when $n = 0$? Show it.

What kind of equation is this when $n = 1$? Show it.

This will turn it into a linear equation.

Rationale and Method for Solving Bernoulli Equations :

For values other than 0 or 1, we will use the substitution $v = y^{1-n}$.

We will start by dividing the original equation $\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n$ by y^n. This gets us $y^{-n}\frac{dy}{dx} + P(x)\cdot y^{1-n} = Q(x)$. When we take $v = y^{1-n}$, we see that $\frac{dv}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$. Rewriting this let's us see that $y^{-n}\frac{dy}{dx} = (1-n)^{-1}\frac{dv}{dx}$.

This makes our equation (which was $y^{-n}\frac{dy}{dx} + P(x)\cdot y^{1-n} = Q(x)$ as you will remember) now $(1-n)^{-1}\frac{dv}{dx} + P(x)\cdot v = Q(x)$. Now, this $(1-n)^{-1}$ is a real number. So multiply the equation by $(1-n)$ and it will yield $\frac{dv}{dx} + (1-n)P(x)\cdot v = (1-n)Q(x)$.

Do you see this as a linear equation? However, you will notice that we need to redefine the $P(x)$ and $Q(x)$ of the generic linear formula, known to us as $\frac{dy}{dx} + P(x)\cdot y = Q(x)$.

expl 4: Solve by the method of Bernoulli equations.

$$\frac{dy}{dx}+\frac{y}{x-2}=5\left(x-2\right)y^{1/2}$$

Compare to
$\dfrac{dy}{dx}+P(x)\cdot y=Q(x)\cdot y^{n}$.
What is $P(x)$, $Q(x)$, and n?

What do we divide by? What do we let v be? Work to get in the form $\dfrac{dv}{dx}+P(x)\cdot v=Q(x)$.

More room on next page.

(extra room for work)

Do *not* forget to resubstitute to get v out of the solution.

Additional solutions?

Consider the previous example.

Since the original equation and our modified one differ only by a factor of $y^{1/2}$, they should have the same solutions except when $y = 0$. Notice this is a solution to the original diff. eq. that does *not* show up because we divided out by $y^{1/2}$.

Show that this solution $y = 0$ does make the original equation true while it is *not* a solution to the modified equation we actually solved.

Type 4: Equations with Linear Coefficients:

In some cases, we must transform both x and y into new variables, say u and v. This is true of differential equations with linear coefficients in the form
$\left(a_1x+b_1y+c_1\right)dx+\left(a_2x+b_2y+c_2\right)dy=0$ where $a_i,\, b_i,\, c_i \in \mathbb{R}$.

Depending on the values of the coefficients, we see four subtypes.

Subtype 1: When $a_1b_2 = a_2b_1$, the equation can be put into the form $\dfrac{dy}{dx}=G(ax+by)$ which can be solved by substitution.

Subtype 2: When $a_2 = b_1$, the equation is exact.

Subtype 3: When $c_1 = c_2 = 0$, the equation is homogeneous and can be written as

$\dfrac{dy}{dx}=-\dfrac{a_1x+b_1y}{a_2x+b_2y}=-\dfrac{\left(a_1x+b_1y\right)\left(1/x\right)}{\left(a_2x+b_2y\right)\left(1/x\right)}=-\dfrac{a_1+b_1\left(y/x\right)}{a_2+b_2\left(y/x\right)}$. Again, this is homogeneous, isn't it?

Subtype 4: When $a_2 \neq b_1$ and $a_1b_2 \neq a_2b_1$, then we must find a translation of axes in the form $x = u + h$ and $y = v + k$, $h,k \in \mathbb{R}$, that will make $a_1x+b_1y+c_1=a_1u+b_1v$ and $a_2x+b_2y+c_2=a_2u+b_2v$.

Such a transformation *exists if* the system of equations $a_1h+b_1k+c_1=0$, $a_2h+b_2k+c_2=0$ has a solution. A solution is ensured by the fact that $a_1b_2 \neq a_2b_1$.

So what we will do is solve the system to find h and k. Then we will use the substitutions $x = u + h$ and $y = v + k$ to solve the homogeneous equation $\dfrac{dv}{du}=-\dfrac{a_1u+b_1v}{a_2u+b_2v}=-\dfrac{a_1+b_1\left(v/u\right)}{a_2+b_2\left(v/u\right)}$.

You can substitute for x and y into the original equation and solve for this each time. However, it is easier to use this as a formula.

expl 5: Solve by the method for equations with linear coefficients.
$\left(-4x-y-1\right)dx+\left(x+y+3\right)dy=0$

Identify $a_1x+b_1y+c_1$ and $a_2x+b_2y+c_2$. Which of the four subtypes do we have?

Find h and k. We will use substitutions.

Although we can use the “formula” on page 9 to rewrite the equation in terms of u and v, let’s do it from scratch. This will make us much, much better people; aaaah.

(extra room for work)

Our equation will now be homogeneous. Use the substitution $z = {}^{v}\!/_{u}$. We now have the equation $\dfrac{dv}{du} = G(z)$.

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{u+a}{u-a}\right| + c$$

(Notice how u and a changed position!)

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$|a|\cdot|b| = |ab|$$

$$|z-2| = |2-z|$$

$$\log_a x + \log_a y = \log_a (x \cdot y)$$

$$\log_a x - \log_a y = \log_a \left({}^{x}\!/_{y}\right)$$

(extra room for work)

Remember to resubstitute back to the original variables.

This was separable, so check $G(z) - z = 0$ for additional solutions.

Be sure to write your final solution, considering the additional solutions found above.

Glory, Glory, Algebra!

Worksheet: Crazy differential equations for you to solve:

That sums it up. One problem will ask you to categorize an equation into the various types we see in this section. The second problem will give you an equation in the form $\frac{dy}{dx} = G(ax + by)$ to solve.