Math 127
Section 10.5
Partial Fraction Decomposition

We will do #30 and #24 from the homework section of 10.5 on page 815.

#30

Write the partial fraction decomposition of $\dfrac{x^2 - x - 8}{(x+1)(x^2+5x+6)}$**.**

We look at the denominator (x + 1)(x^2 + 5x + 6). The question is whether or not the quadratic factor is irreducible. Considering x^2 + 5x + 6, notice b^2 - 4ac is <u>not</u> negative so there are real numbers x_1 and x_2 such that x^2 + 5x + 6 = (x – x_1) (x – x_2). Factor x^2 + 5x + 6 to find (x – x_1) (x – x_2).

This means that the whole denominator is factored as ???

So the denominator of our rational function factors into three non-repeated linear factors. This falls under case one on page 809-810. Hence

$$\frac{x^2 - x - 8}{(x+1)(x+3)(x+2)} = \frac{A_1}{(x+1)} + \frac{A_2}{(x+3)} + \frac{A_3}{(x+2)} \;.$$

To find A_1, A_2, and A_3 multiply both sides of this equation by the least common denominator. (We will create three equations with three unknowns, which we will then solve.)

We get $x^2 - x - 8 = A_1(x+3)(x+2) + A_2(x+1)(x+2) + A_3(x+1)(x+3)$.
Multiply the factors together on the right side.

We get $x^2 - x - 8 = A_1\left(x^2+5x+6\right)+A_2\left(x^2+3x+2\right)+A_3\left(x^2+4x+3\right)$.

Notice we can rearrange this into
$x^2 - x - 8 = \left(A_1+A_2+A_3\right)x^2+\left(5A_1+3A_2+4A_3\right)x+\left(6A_1+2A_2+3A_3\right)$.

This means $(A_1 + A_2 + A_3)\, x^2 = 1\, x^2$. We also get $\left(5A_1+3A_2+4A_3\right)x = -x$ and $\left(6A_1+2A_2+3A_3\right) = -8$.

Dividing the first two equations by x^2 and x respectively,
we see that $A_1 + A_2 + A_3 = 1$.
Likewise, $5A_1 + 3A_2 + 4A_3 = -1$
and $6A_1 + 2A_2 + 3A_3 = -8$.

Notice this is a system of three equations in three variables. Form a 3x4 matrix with these equations and use the calculator to solve them. What are the values of A_1, A_2, and A_3?

This means $\dfrac{x^2 - x - 8}{(x+1)(x+3)(x+2)} = \dfrac{-3}{(x+1)}+\dfrac{2}{(x+3)}+\dfrac{2}{(x+2)}$. You could check this result by adding the fractions on the right to see if the sum is indeed equal to the left.

#24

Write the partial fraction decomposition of $\dfrac{10x^2+2x}{(x-1)^2(x^2+2)}$.

Notice the denominator contains an irreducible quadratic factor (x^2 + 2) and a repeated linear factor (x – 1)^2. This falls under case 2 (pg 811) and case 3 (pg 813).

Hence the decomposition of $\dfrac{10x^2+2x}{(x-1)^2(x^2+2)}$ has the terms $\dfrac{A_1}{(x-1)}$ and $\dfrac{A_2}{(x-1)^2}$. This takes care of the (x – 1)^2 part as per case 2. (A note for understanding: If the denominator contained (x – 1)^3 instead of (x – 1)^2, we would also have the term $\dfrac{A_3}{(x-1)^3}$. However, this is not the case.)

The decomposition also has the term $\dfrac{A_3x+B}{(x^2+2)}$. This takes care of the (x^2 + 2) part as per case 3.

So we have $\dfrac{10x^2+2x}{(x-1)^2(x^2+2)} = \dfrac{A_1}{(x-1)} + \dfrac{A_2}{(x-1)^2} + \dfrac{A_3x+B}{(x^2+2)}$. Multiply by the least common denominator (x – 1)^2(x^2 + 2) to start solving for A_1, A_2, A_3, and B.

We get $10x^2 + 2x = A_1(x-1)(x^2+2) + A_2(x^2+2) + (A_3x+B)(x-1)^2$. Multiply the factors together on the right side.

Now organize the x^3 terms so they are together. Do the same for the x^2 terms, the x terms, and the constant terms.

This leads us to

$$\begin{array}{lllll} A_1 & & + A_3 & & = 0 \\ -A_1 & + A_2 & - 2A_3 & + B & = 10 \\ 2A_1 & & + A_3 & - 2B & = 2 \\ -2A_1 & + 2A_2 & & + B & = 0 \end{array}$$

Use your calculator to solve this system. Put it in as a 4x5 matrix and perform RREF.

This should give you $A_1 = \frac{14}{3}$, $A_2 = 4$, $A_3 = \frac{-14}{3}$, and $B = \frac{4}{3}$.

This means that $\frac{10x^2 + 2x}{(x-1)^2(x^2+2)} = \frac{14}{3(x-1)} + \frac{4}{(x-1)^2} + \frac{-14x+4}{3(x^2+2)}$.