Math 119

Main Points of Discussion

1. Solving equations:

When you have an equation like $y = 2x^2 - 3x + 5$, you should see a relationship between two variables, x and y. The graph of $y = 2x^2 - 3x + 5$ is the picture of this relationship. Solving an equation like $9 = 2x^2 - 3x + 5$ means you're finding the x (or x's) that is (are) paired with the y value of 9. So it works to graph both $y = 2x^2 - 3x + 5$ and $y = 9$ and see where they intersect. The points of intersection give the two x values that make $2x^2 - 3x + 5$ equal to 9. Notice the two values -.85 and 2.35 make the equation true.

2. Rules of Exponents:

Do not try to memorize them. To remember the rule $\frac{x^n}{x^m} = x^{n-m}$, I simply think about a problem like $\frac{x^5}{x^2} = \frac{xxxxx}{xx} = \frac{(xx)xxx}{(xx)} = xxx = x^3 = x^{5-2}$ because the xx is a factor on top and bottom and therefore reduces to 1. To remember the rule $\left(x^n\right)^m = x^{nm}$, I do quick problems like $(2^2)^3 = (2^2)(2^2)(2^2) = 2*2*2*2*2*2 = 2^6 = 2^{2*3}$. Try some out now!

3. Solving equations by finding x-intercepts:

To solve equations in the form $f(x) = 0$, graph $y = f(x)$ and then look for where y is zero. These are the x-intercepts of the graph of $y = f(x)$. Remember to use the Root or Zero Function on your calculator to get an accurate answer as opposed to just using the Trace button. For instance, to solve $4x^2 + 5x - 8 = 0$, graph $y = 4x^2 + 5x - 8$ and see where y is zero (the x-intercepts).

4. Finding *x* and *y* intercepts algebraically:

To find the y-intercept, imagine the graph going through the y-axis. The one thing you know is that the x value of this point must be zero (because all points on the y-axis have an x value of zero). So we substitute zero for x in the given equation and solve for y.

The same reasoning is used for the x-intercept; imagine the graph going through the x-axis. The one thing you know is that the y value of this point must be zero (because all points on the x-axis have a y value of zero). So we substitute zero for y in the given equation and solve for x.

5. Quadratic Formula Program QUADRATC or QUAD2:

This will solve quadratic equations that have zero on one side like $3x^2 - 4x - 5 = 0$. The calculator program will give you decimal answers but sometimes the answers (in the book) are written in the form of $\frac{4 \pm \sqrt{76}}{6}$. On the final exam, they will be in decimal form, rounded to two decimal places.

As I said, sometimes books write the answers in radical form. To figure out if their answer is the same as your solution, you must figure the decimal equivalent to $\frac{4 \pm \sqrt{76}}{6}$ on your calculator. To do this, enter in $(4+\sqrt{76})/6$ then hit ENTER. This gives you the decimal equivalent to the first one. (If you have a TI-83 and possibly other calculators, when you hit the square root symbol, it puts a parenthesis with it automatically, your screen will really look more like this: $\left(4+\sqrt{}(76)\right)/6$. Notice the extra parenthesis needed after the 76.) You will see this is equal to 2.12.

Then hit **2nd ENTER** to make the calculator repeat what you just entered, change the plus sign to a minus, then hit **ENTER**. This will give you the second root in decimal form, -.79. This will allow you to compare the radical answers and the answers your calculator program gives you.

6. Complex solutions to quadratic equations:

Recall how the solutions for quadratic equations like $3x^2 + 1x + 6 = 0$ are the x-intercepts of the relationship $y = 3x^2 + 1x + 6$. But notice when you graph $y = 3x^2 + 1x + 6$, you see there are no x-intercepts. That means that there are no real numbers that make the equation $3x^2 + 1x + 6 = 0$ true. We have to reach outside the real number realm to find solutions. This is where the complex numbers come into play. Recall, by the quadratic formula, the solutions would be

$x = \dfrac{-1 \pm \sqrt{1^2 - 4(3)(6)}}{2(3)} = \dfrac{-1 \pm \sqrt{-71}}{6} = \dfrac{-1 \pm \sqrt{71} * i}{6} = -.17 \pm 1.40i$. These are the two (complex) numbers that make the equation $3x^2 + 1x + 6 = 0$ true. Your calculator program will give you complex solutions when they are appropriate.

7. Functions and functional notation:

If for each x value there is only one y value, then the relationship between x and y is a function. (This is what the Vertical Line Test actually tests.) This is important because functions are unambiguous. Everyone gets the same y value out for a given x. That's why functions are very important to us algebraists. Sometimes it is helpful to see functions as machines that act on numbers, producing outputs. Other times it is helpful to think of functions as rules, or simply relationships.

Functional notation helps us see functions as rules that tell us what to do to an x value to get a y value out. For instance, the function $f(x) = 5x^2 - 8$ is a rule that tells us to square the x, multiply by 5, and then subtract 8 to get the y or $f(x)$ value. Whatever is in the parentheses, we'll do that to it. So to find $f(2)$, we square 2, multiply that by 5, and then subtract 8. This gets us $f(2) = 12$.

Do the same for $f(x-3)$. We see we get

$f(x-3) = 5(x-3)^2 - 8 = 5(x^2 - 6x + 9) - 8 = 5x^2 - 30x + 45 - 8 = 5x^2 - 30x + 37$. Try to find $f(x+2)$ for $f(x) = 5x^2 - 8$.

8. Lines:

Finding a line's equation means finding how x and y are related. You want to find an equation in the form $y = mx + b$ where m is the slope of the line and b is the y-intercept. Notice this equation (like $y = 2x + 4$) tells you how x and y are related (you multiply x by 2 and add 4 to get y). So to make the equation $y = mx + b$, you need to find m and b.

Let's say you are given the problem "Given the two points (2, 3) and (4, 2), find the equation of the line through these points." First, find the slope. Do it now.

Did you get $-1/2$? So you know x and y are related in the fashion $y = \frac{-1}{2}x + b$ where you must now find b. But you also know that these two points (2, 3) and (4, 2) are related by this equaiton $y = \frac{-1}{2}x + b$. So substitute the coordinates of one of the points into the equation for x and y to find b. Do it now.

This should get you $b = 4$. So the line's equation is $y = \frac{-1}{2}x + 4$. Then check your equation by making sure each point satisfies it.

9. Story Problems:

Let x represent the quantity you want to find. Make sure you write down specifically how you define your variables. Then form a verbal model using the context of the situation. This takes practice to do successfully so do not give up easily. Then, using your variables, turn your verbal model into an equation to solve. Try solving graphically if it's really complicated.

10. Roots:

Remember if $x^2 = 4$, then $x = \pm\sqrt{4} = \pm 2$ because both 2 squared and –2 squared are 4. (A discussion explaining the $\pm$ part is given in The Basics Part II.) But if $x^3 = 8$, we get only $x = 2$ because 2 cubed is 8, but notice –2 cubed is not 8. Remember the square root of a negative number is a complex number, like $\sqrt{-71} = \sqrt{71} * i = 8.43i$.

11. Inequalities:

As a general rule, all non-linear inequalities are solved graphically. (If the problem is something like $5(x+3) - 4x < 6x + 4$, it is relatively painless to solve algebraically. If there are squared or cubed terms, do not attempt an algebraic solution.) The easiest way to deal with inequalities is to get zero on one side of the inequality, usually by subtracting. Let's say we want to (graphically) solve $\frac{2x^2+4}{5x} \le 2$. Subtract 2 from both sides to get $\frac{2x^2+4}{5x} - 2 \le 0$.

Then graph the left side and write down the x values that produce points on the graph that are below or on the x-axis. You are looking for points below the x-axis because of the "less than" sign. You are also looking for points on the x-axis because it's a "less than *or equal to*" sign.

We usually write solutions to inequalities in interval notation.

12. Interval Notation:

The notation (2, 10) means the interval of real numbers between 2 and 10, but not including 2 or 10.

The notation [2, 10) means the interval of real numbers between 2 and 10, including 2, but not including 10.

The notation [2, 10] means the interval of real numbers between 2 and 10, including both 2 and 10.

The notation $(-\infty, 0]$ means the interval of real numbers less than or equal to zero. Notice when we have the infinity symbol in interval notation, we use a parenthesis.

I remember the difference between parentheses and brackets by thinking that brackets are hard and rigid and hold the number in, whereas parentheses are loose and let the number quietly slip out, but the numbers up to that number are still held in.

13. End behavior of Polynomials:

The leading coefficient's sign and the degree of the polynomial determine a polynomial's end behavior. To remember the rules, I keep a picture of $y = x^2$ and $y = x^3$ in my head. Notice $y = x^2$ has the end behavior "as $x \to -\infty, y \to \infty$ and as $x \to \infty, y \to \infty$".

This helps me remember that all polynomials that are of even degree and have a positive leading coefficient have this same end behavior.

Similarly, notice $y = x^3$ has the end behavior "as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$". This helps me remember that all polynomials that are of odd degree and have a positive leading coefficient have this same end behavior.

14. Vertical asymptotes of rational functions:

Remember the vertical asymptotes occur because the function is undefined. This happens when the bottom of a fraction is zero. So to find vertical asymptotes, you find which x values make the bottom zero. This does not work if there are common factors between the top and the bottom of the rational function. Remember here, there may be holes in the graph instead of vertical asymptotes. You may want to write an example of this now.

15. Extraneous solutions:

When you solve any equation, plug your answers back into the original equation to check. There are a few techniques we use that will create extraneous solutions and you **must** check those solutions. (These techniques include squaring both sides of an equation and solving logarithmic equations.) But really it's a good idea to always check your solution. So get into the habit.

16. Inverse functions:

Remember inverse functions undo each other. To find an inverse graphically, we switch the x's and the y's of the points. This coincides with the idea of how we find an inverse algebraically, given an equation of a function. Let's say you want to find the inverse of $y = 3x + 2$.

First, switch the x's and the y's to get $x = 3y + 2$.

Second, solve for y, since we usually write equations with the y isolated. Do this now.

To check that two functions $f(x)$ and $g(x)$ are inverses, you show that $f(g(x)) = x$ and also that $g(f(x)) = x$. Notice this is the same as saying "f undoes g" and also "g undoes f".

Let $f(x) = 3x + 2$ and $g(x) = \dfrac{x-2}{3}$. You find $f(g(x))$ in the following way.

$$f(g(x)) = f\left(\frac{x-2}{3}\right) = 3\left(\frac{x-2}{3}\right) + 2 = x - 2 + 2 = x$$

You show $g(f(x)) = x$ similarly. Composition was covered in Advanced Functions and Equations I and is essential to understand this. Try showing that $g(f(x)) = x$ now.

17. Inverse relationship between logs and exponentials:

Know that $y = b^x$ and $y = \log_b x$ are inverses. We can see this by interpreting $y = \log_b x$ as "y is the number to which I raise b to get x" or "$b^y = x$". Notice $b^y = x$ would be the first step to finding the inverse of $y = b^x$ algebraically.

The fact that the exponential function and the logarithmic function are inverses makes it possible to solve equations that involve the variable x being in the exponent like $2^x = 9$ or inside a log like $\log(x + 2) = 4$.

18. Equivalent relationship between logs and exponentials:

Know that $2^x = N$ and $\log_2 N = x$ are equivalent. Notice we can see this by interpreting $\log_2 N = x$ as "x is the number to which I raise 2 to get N". That is exactly what $2^x = N$ tells us.

This also helps us solve equations that involve the variable x being in the exponent like $2^x = 9$ or inside a log like $\log(x + 2) = 4$ because when we rewrite the equation in the alternate form, we can get at the variable x. Try solving these equations now.

19. Solving equations:

Remember solving an equation can be thought of as undoing what was done to x. For instance, let's solve the equation $8 = 2x + 3$. Notice this equation tells us that some number, call it x, is multiplied by 2 and then we add 3 to get 8. And we want to know what number x this works for.

Well, we could think of solving the equation as uncovering the x by undoing the operations that was done to x, but in reverse order. To do this, first we subtract 3, and then divide by 2. We do this to both sides of the equation to get $\frac{5}{2} = x$.

Now let's bring the idea of functions and inverses into this. We could describe the actions done to x (Remember in the original equation, x is multiplied by 2 and then we add 3 to get 8) as the function $f(t) = 2t + 3$. We could describe the operations we do to solve the equation as $g(t) = \frac{t-3}{2}$. Notice these functions f and g are inverses.

This is the point! When you solve equations, what you are really doing is using inverses and functions. (By the way, this only works because we are dealing with functions. Do you know why?) This is the same reason we also use the inverse relationship between logarithmic and exponential functions to solve those types of equations.

20. Compound interest:

Compound interest is an application of the exponential function. Recall that the setup is that we invest a certain amount in the bank, call it our principal or P. We earn interest on this money at a rate of r (decimal form, 7% is .07) for a time period of t years. Every so often, say n times a year, the interest is added to our principal and after that, we earn interest on the previous interest as well as the initial investment. This means the interest is compounding.

The formula for the amount of money A in the bank after t years is $A = P\left(1+\frac{r}{n}\right)^{nt}$. We will use this formula to answer various questions concerning investments or loans. When the problem says it is compounded continuously, we use the formula $A = Pe^{rt}$ where e is that irrational number approximately equal to 2.72. The worksheet "Exponential/Logarithmic applications: Compound interest" tries to show you how these two formulas are related and provide practice problems for using both.

21. Lovely, lovely algebra:

Remember algebra is the art of using the information you know to find the information you want to know. Using careful notation and understanding what you've written will get you 75% of the way there. Good luck on your Final Exam.