We turn from local extrema to absolute extrema.

Business Calculus
Class notes
Section 3.4: Applications of Derivatives: Optimization (Absolute Maximums and Minimums)

We might have a function that describes the revenue of a company and be interested in maximizing that revenue. Say you know the trajectory of a near-Earth object and want to know its minimum distance to our planet. We start by defining absolute (a.k.a. global) maximums and minimums as seen back in algebra. The set D is the entire **domain** of the function.

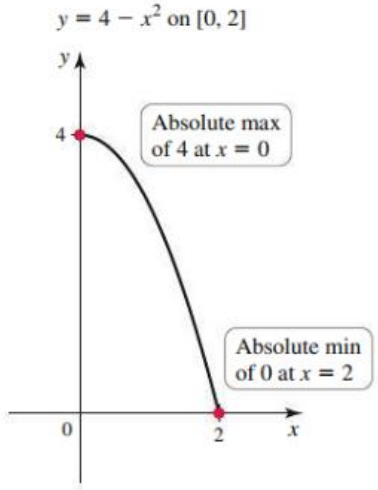
DEFINITION Absolute Maximum and Minimum

Let f be defined on a set D containing c. If $f(c) \geq f(x)$ for every x in D, then $f(c)$ is an **absolute maximum** value of f on D. If $f(c) \leq f(x)$ for every x in D, then $f(c)$ is an **absolute minimum** value of f on D. An **absolute extreme value** is either an absolute maximum value or an absolute minimum value.

The graph shown here has both an absolute max and absolute min. Do you see why the definition above describes the points?

Singular forms: Minimum, Maximum, Extremum

Plural forms: Minima, Maxima, Extrema



If we are asked, "Where does the extremum occur?", we give the x-value or $x = c$.

If we are asked, "What is the extremum?", we give the y-value or $f(c)$.

We often state this as, "The extremum is _____ at _____."
(y-value) (x-value)

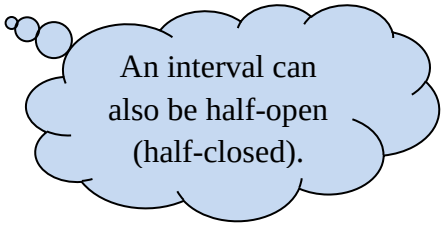
We have explored how these points *may* be where $f'(c) = 0$. You should see this for the absolute maximum above but *not* the minimum. (The minimum, in fact, occurs at the endpoint $x = 2$ but $f'(2) \neq 0$.) We have much more to investigate.

Finding Absolute Extrema Over Closed Intervals:
If we have a function and are asked to find its extrema, it's nice to know when it is guaranteed to exist. We have this piece of the puzzle.

> **THEOREM 4.1 Extreme Value Theorem**
> A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.

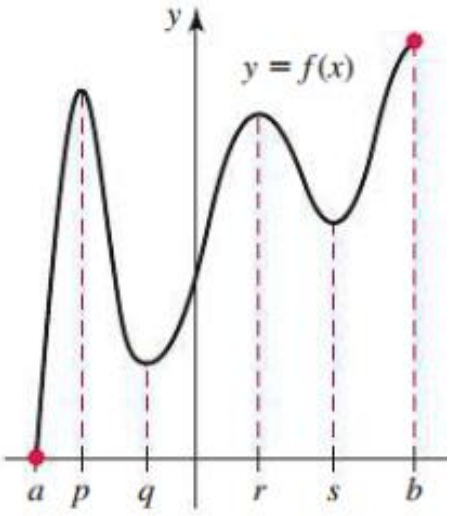
Recall: Definition: Closed Interval: A closed interval is one where the endpoints are included in the interval such as $[0, 1]$ or $[a, b]$.

Recall: Definition: Open Interval: An open interval is one where the endpoints are *not* included in the interval such as $(0, 2)$ or (a, b).



expl 1: The function pictured here is defined on the closed interval $[a, b]$.
a.) Is the function continuous on $[a, b]$?

b.) Does the Extreme Value Theorem apply to this function? If so, find the absolute max and min. Where do the extrema occur?



We will use the procedure on the following page. It is called Theorem 8 in our book.

Theorem 8: Procedure for Finding Absolute Extrema Over a Closed Interval:

Maximum–Minimum Principle 1

Suppose f is a continuous function defined over a closed interval $[a, b]$. To find the absolute maximum and minimum values over $[a, b]$:

We must be given a closed interval.

a. Find $f'(x)$.

b. Determine all critical values in $[a, b]$. That is, find all c in $[a, b]$ for which

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

c. List the values from step (b) along with the endpoints of the interval:

$$a,\ c_1,\ c_2,\ \ldots,\ c_n,\ b.$$

d. Evaluate $f(x)$ for each value in step (c):

$$f(a),\ f(c_1),\ f(c_2),\ \ldots,\ f(c_n),\ f(b).$$

The largest of these is the **absolute maximum of f over** $[a, b]$. The smallest of these is the **absolute minimum** of f over $[a, b]$.

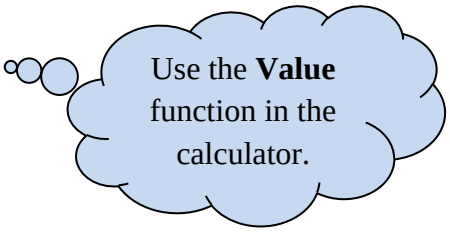
expl 2: Conduct this procedure to find any absolute extrema over the interval [-1, 2].

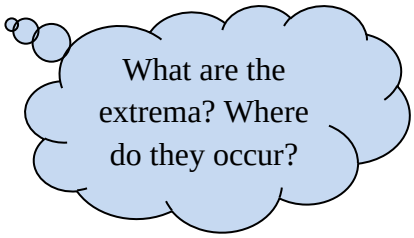
$y = \ln\left(x^3 - 2x + 8\right)$

Since we know $x^3 - 2x + 8 > 0$ (why?), we do *not* have to look for where $f'(x)$ is undefined. Do you see why?

Continue on next page.

More room for example 2:





expl 3: Conduct this procedure to find any absolute extrema over the interval [-4, 5].

$h(x) = (x+3)^{2/3} - 5$

Connection Between Local and Absolute Extrema:

If the conditions hold for the principle below, then we may be able to shortcut the process (a bit). Notice this does *not* require a closed interval. However, this does require the second derivative is *not* zero. If it is zero, we must fall back to the First Derivative Test.

Maximum–Minimum Principle 2

Suppose f is a function such that $f'(x)$ exists for every x in an interval I and there is *exactly one* (critical) value c in I, for which $f'(c) = 0$. Then

$f(c)$ is the absolute maximum value over I if $f''(c) < 0$

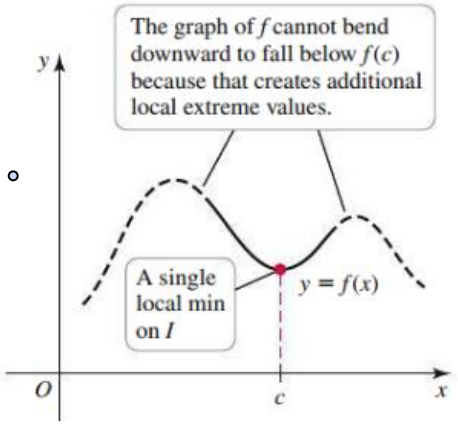
or

$f(c)$ is the absolute minimum value over I if $f''(c) > 0$.

The Second Derivative Test gets us that $f(c)$ is a local extremum. This goes further if an interval contains *only one* such value of c.

Here is a good example of why this theorem must be true. Would you agree with it?

Check the conditions listed above.



expl 4: Find the absolute extremum for the following function on the interval (-∞, ∞).

$y = 2xe^{-6x}$

We *cannot* use the first principle as this interval is *not* closed.

Continue on next page.

More room for example 4:

