Oy, oy, oy… What to say here?
Lots to do. Let's get to it.

Business Calculus
Class Notes
Section 3.6: Applications of Derivatives: Optimization (Marginals, Differentials, and Linearization)

Marginal Cost, Revenue, and Profit:
Suppose we have functions that define the revenue, cost, and (their difference) profit for a company when they produce and sell x items. How can calculus help us learn more about the company's financial situation?

The derivative, of course! Here is how the book puts it.

Let $C(x)$, $R(x)$, and $P(x)$ represent, respectively, the total cost, total revenue, and total profit for the production and sale of x items.

The **marginal cost** at x, given by $C'(x)$, is the approximate cost of the $(x+1)^{st}$ item:

$$C'(x) \approx C(x+1) - C(x)$$

and the cost of producing $x+1$ items is approximated by

$$C(x+1) \approx C(x) + C'(x).$$

This should be reminiscent of slope as it is the derivative. Notice the *approximately* equal sign.

The **marginal revenue** at x, given by $R'(x)$, is the approximate revenue from the $(x+1)^{st}$ item:

$$R'(x) \approx R(x+1) - R(x)$$

and the revenue from selling $x+1$ items is approximated by

$$R(x+1) \approx R(x) + R'(x).$$

Perhaps we do *not* know the revenue function $R(x)$. However, we may know $R(a)$ and $R'(a)$ for a particular value of x.

The book covers marginal profit on the next page.

The **marginal profit** at x, given by $P'(x)$, is the approximate profit from the $(x+1)^{st}$ item:

$$P'(x) \approx P(x+1) - P(x)$$

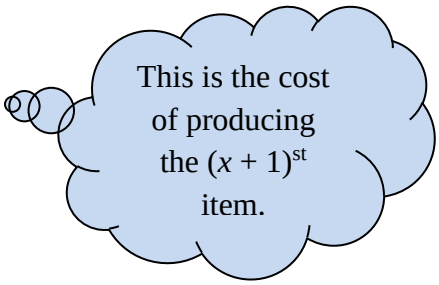
and the profit from production and sale of $x+1$ items is approximated by

$$P(x+1) \approx P(x) + P'(x).$$

expl 1: A company knows that the daily cost, in dollars, of producing x backpacks is given by $C(x) = 0.001x^3 + 0.07x^2 + 19x + 700$. Answer the following questions.

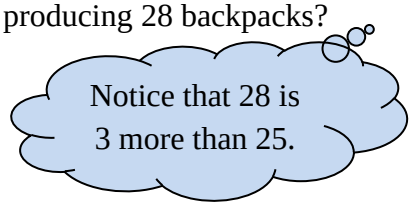
a.) If the current production is 25 backpacks, find the current cost.

b.) Find a formula for the marginal cost.



c.) Use the formula $C(x+1) \approx C(x) + C'(x)$ to estimate the cost of producing 26 backpacks.

d.) How would you amend the formula in part *c* to estimate the cost of producing 28 backpacks? Do it and find the value.



Differentials and Delta Notation:

Think back to the difference quotient that we used to explore a function's rate of change. This quickly morphed into the derivative of the function.

Since h was the amount of change from one x-value to another on that secant line (which is $(x + h) - x$), we could think of this difference as Δx.

This is the Greek letter delta and can be read as "change in".

Using this new symbol, we could write this slope as $m_{\text{sec}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$. Since this secant line's slope *approximates* the tangent line's slope (and that's the derivative $f'(x)$), we could write $f'(x) \approx \frac{\Delta y}{\Delta x}$. Multiplying both sides by Δx yields $\Delta y \approx f'(x) \cdot \Delta x$. This will be true of continuous, differentiable functions and small values of Δx.

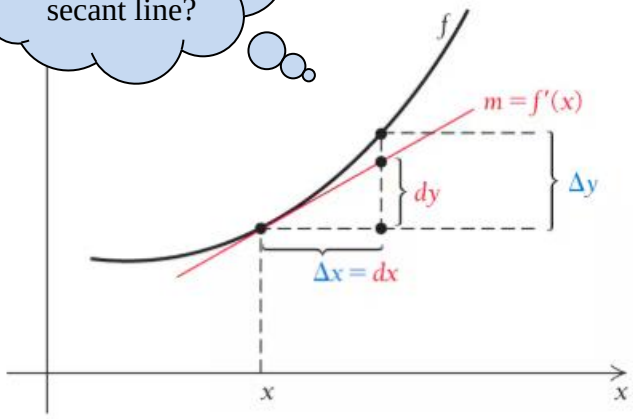
As illustrated in this picture, we will use ***dx*** to represent Δx and ***dy*** to *estimate* Δy.

These are called **differentials**.

Definition: Differential, *dx*: This is equal to Δx.

Definition: Differential, *dy*: From the picture, notice that $dy \approx \Delta y$. Also from the picture, notice that $f'(x) = \frac{dy}{dx}$ as it's the slope of this tangent line.

Can you draw in the secant line?



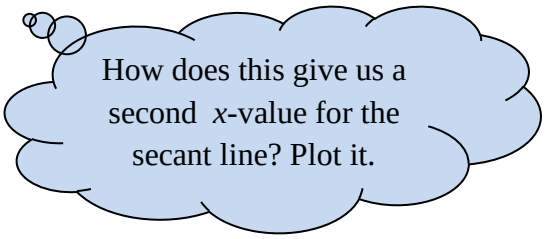
Recall, this alternative notation for the derivative was given in an earlier section. Here, we see its true meaning and acknowledge that dy and dx are entities in their own right.

Seeing these differentials as individual pieces in their own right lets us write $dy = f'(x)\,dx$. This type of rewrite is useful often. Consider the surface area function (for half of a sphere) $A(r) = 2\pi r^2$ taking r to be its radius. We can write $dA = A'(r)dr$ and use it to explore how area changes with respect to the change in its radius.

Homework tip!

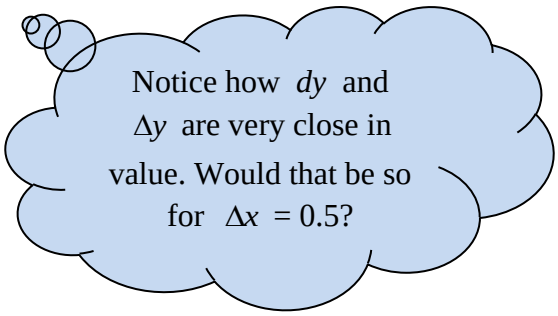
expl 2: Consider the function $f(x) = x^3$.
Focus on $x = 2$ and use 0.01 for Δx.
Draw a quick, zoomed-in graph for the function. Label the point (2, 8). Complete the following.

a.) Find Δy given that $x = 2$ and $\Delta x = 0.01$.



b.) Use $dy = f'(x)dx$ to write a formula for dy. (Leave dx in place.)

c.) Use the formula from part *b* to find dy for the specific x-value and Δx given.



expl 3: Use $\Delta y \approx f'(x)\cdot\Delta x$ to find a decimal approximation to $\sqrt{123}$. Round to the nearest thousandths. Select a convenient y-value (what perfect squares do you know?) and then think of what $f(x)$ and Δx need to be.

Keep in mind that 123 is an x-value.

Once you have Δy, how do get the approximation for $\sqrt{123}$?

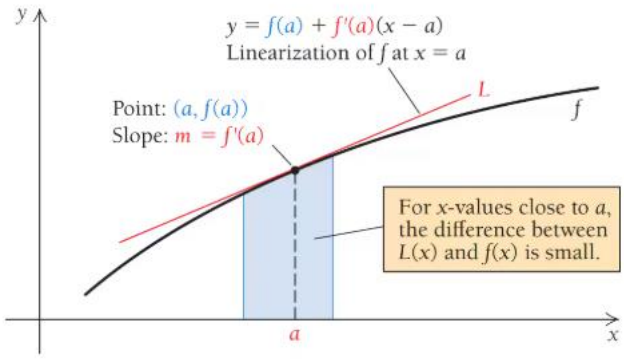
Linearization:

You might recall that we have used the formula $y - f(a) = f'(a)(x-a)$ to find the **equation of the tangent line** at some x-value $x = a$. Solve this for y and you get $y = f(a) + f'(a)(x-a)$.

Here's the book's picture.

You will notice that this line could serve as a good approximation of the function value for some close-by x-value. That is, if you stay relatively close to the original x-value a.

This is often referred to as $L(x)$ for the **linearization of the function $f(x)$**.



Of course, for this to work, the function f has to be differentiable at $x = a$ (or rather, $f'(a)$ must exist).

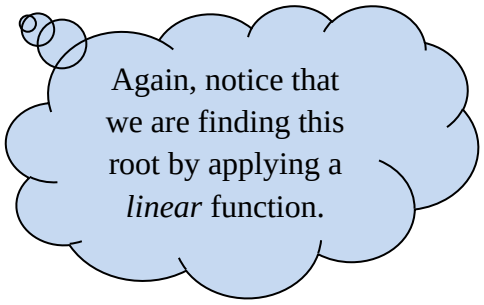
So, we have $L(x) = f(a) + f'(a)(x-a)$.

If you think about it, this turns finding a function value for any crazy function into just a linear calculation.

expl 4: Let $h(x) = \sqrt{x} = x^{1/2}$. Answer the following.

a.) Find the linearization $L(x)$ of h at $x = 16$. Simplify.

b.) Use your formula for $L(x)$ to approximate $\sqrt{17}$.



c.) Find $\sqrt{17}$ with the calculator and comment on how good the approximation is. Marvel at your genius.