

We have explored limits graphically. Now we see how to find them algebraically (analytically).

Business Calculus

Class Notes

Section 1.2: Algebraic Limits and Continuity

Think about the function $y = 4$. In fact, give yourself a pretty graph.

What would you say $\lim_{x \rightarrow a} 4$ is, no matter what value we use for a ?

There is a rule for that and many more. A little exploration will make these rules (properties) more understandable.

Limit Properties:

If $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$, and c is any constant, then we have the following limit properties.

L and M are real numbers

$\lim_{x \rightarrow a} 4 = 4$ (using graphical method from 1.1)

$c \in \mathbb{R}$

" c is an element of the real numbers"

L1. The limit of a constant function is the constant itself:

$$\lim_{x \rightarrow a} c = c.$$

L2. The limit of a power function is the limit of the base, raised to that power:

$$\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n = L^n, \text{ and } \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L}.$$

In the case of the power, we must have $L \neq 0$ if n is negative, and in the case of the root, we must have $L \geq 0$ if n is an even integer.

Remember that $L^{-n} = \frac{1}{L^n}$ and we cannot divide by 0. Also, we cannot take even roots of negative numbers (in the real numbers).

Let $f(x) = 8x$

We can find

$$\lim_{x \rightarrow 3} (8x) = 24.$$

So, L2 tells us

$$\text{that } \lim_{x \rightarrow 3} (8x)^5 = 24^5$$

OR

$$\lim_{x \rightarrow 3} \sqrt{8x} = \sqrt{24}$$

L3. The limit of a sum or difference is the sum or difference of the limits:

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M.$$

L4. The limit of a product is the product of the limits:

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \left[\lim_{x \rightarrow a} f(x) \right] \cdot \left[\lim_{x \rightarrow a} g(x) \right] = L \cdot M.$$

L5. The limit of a quotient is the quotient of the limits:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}, \text{ assuming } M \neq 0.$$

L6. The limit of a constant times a function is the constant times the limit of the function:

$$c \in \mathbb{R}$$

$$\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x) = c \cdot L.$$

Property L6 combines L1 and L4 but is stated separately because it is used so frequently.

expl 1: Use the rules to say if each statement is true or false. If true, tell which property says so. If false, say why, perhaps by finding the rule that is being misused.

a.) If $\lim_{x \rightarrow 2} f(x) = 9$, then $\lim_{x \rightarrow 2} \sqrt{f(x)} = 3$. True (L2)

b.) $\lim_{x \rightarrow 2} 7 = 2$

False use of L1 (should be $\lim_{x \rightarrow 2} 7 = 7$)

c.) If $\lim_{x \rightarrow 2} f(x) = 5$, then $\lim_{x \rightarrow 2} (3f(x)) = 15$.

True (L4) or (L6)

2

when we let $g(x) = 3$

if we take $c = 3$.

Limits of Polynomial and Rational Functions:

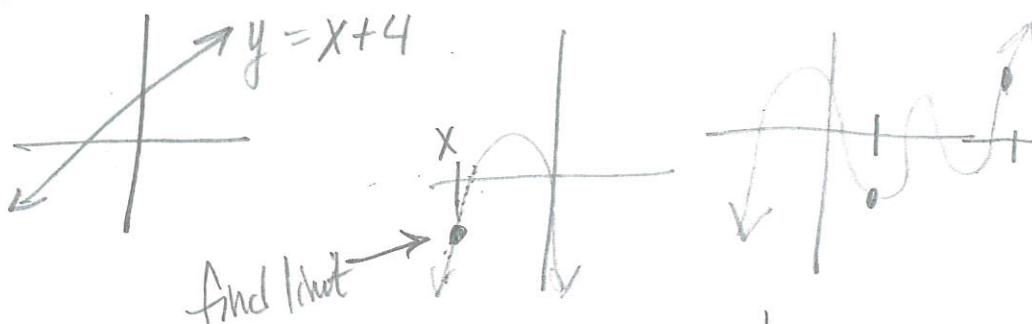
How do we find a limit like $\lim_{x \rightarrow 2} (x^2 - 6x + 9)$? We have a few rules to help us out.

Limits for Polynomial Functions (Linear, Quadratic, Cubic, etc.):

Let $f(x)$ be any polynomial function like the linear $f(x) = mx + b$ or the quadratic $f(x) = ax^2 + bx + c$. (Here, we assume real number coefficients. Recall, a polynomial function is one where the exponents on the x 's are all whole numbers.)

Draw a few examples of polynomial functions here.

whole numbers $\{0, 1, 2, 3, \dots\}$



Notice how they are continuous across their entire domains. **We have the following truth.**

For any polynomial function $f(x)$, we know that $\lim_{x \rightarrow a} f(x) = f(a)$ for any x -value $x = a$.

expl 2: Find $\lim_{x \rightarrow 3} (x^2 + 5)$.

$$= 3^2 + 5$$

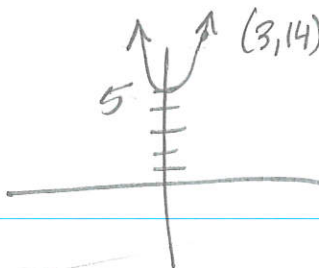
$$= 14$$

$$f(x) = x^2 + 5$$

$$a = 3$$

This is called **direct substitution** and is used often. Our book calls it **direct evaluation**.

Could have graphed



$$y = x^2 + 5$$

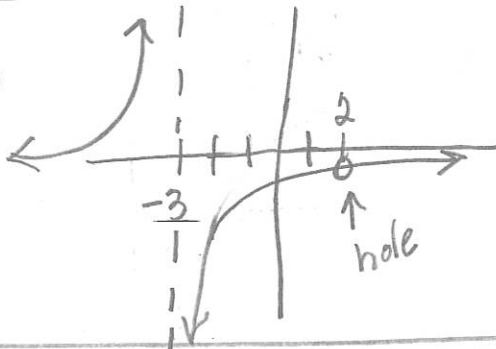
Do you remember what a rational function is? I think of it as a fraction where both the top and bottom are polynomials.

Limits for Rational Functions (Fraction with Polynomials on Top and Bottom):

Let $\frac{p(x)}{q(x)}$ be a rational function. (Here, $p(x)$ and $q(x)$ are known to be polynomials.)

Recall, this rational function is defined for all values of x except when $q(x)$ is zero.

Recall the shape of a rational function and draw one here. Give it a vertical asymptote at $x = -3$ and a hole at $x = 2$. (There are many correct graphs.)



Can you come up with a likely formula?

$$y = \frac{(x-2)}{(x+3)(x-2)}$$

As we see, there may be points of discontinuity on a rational function's graph. We have this truth. We know that $\lim_{x \rightarrow a} \left(\frac{p(x)}{q(x)} \right) = \frac{p(a)}{q(a)}$ except where $q(a)$ is zero.

We will be working with limits a lot as we travel through calculus and beyond. The properties listed earlier will be important.

There's direct substitution again. This is **Theorem 2** in our book.

Handout: Algebra and Geometry Review:

This will come from the front cover of the book. You can also find these online, of course.

expl 3: Find the limit. If it does *not* exist, say so.

$$\lim_{x \rightarrow 3} \left(\frac{x^2 - 25}{x^2 - 5} \right) \quad \begin{array}{l} p(x) = x^2 - 25 \\ q(x) = x^2 - 5 \end{array}$$

$$a = 3$$

$$= \frac{3^2 - 25}{3^2 - 5}$$

$$= \frac{-16}{4}$$

$$= \boxed{-4}$$

Abbreviate "does not exist" as dne.

Check if the value for a is in the domain of the function.

Recall: Exponent rules:

There are many exponent rules you will need to call to mind. A couple of important ones now are $a^{1/2} = \sqrt{a}$ and the more general $a^{1/n} = \sqrt[n]{a}$. ★

expl 4: Find the limit. If it does *not* exist, say so.

$$\lim_{x \rightarrow 4} \sqrt{x^2 - 9}$$

Use L2

$$f(x) = x^2 - 9$$

$$\lim_{x \rightarrow 4} (x^2 - 9) = 4^2 - 9 = 7$$

because of poly func rule on pg 3.

$$\text{Now, } \lim_{x \rightarrow 4} \sqrt{x^2 - 9} = \lim_{x \rightarrow 4} (x^2 - 9)^{1/2} = \left(\lim_{x \rightarrow 4} (x^2 - 9) \right)^{1/2} = 7^{1/2}$$

Theorem 2 does *not* apply. Why? What other property must we use?

not a rational func.

Limits for Rational Functions where a is *not* in the domain:

For these limits, we will use some algebra. Factoring plays a big role. Failing that, we can rely back on numerical or graphical methods.

expl 5: Find the limit. If it does *not* exist, say so.

$$\lim_{x \rightarrow 5} \left(\frac{x^2 - 25}{x - 5} \right) \leftarrow 5 \text{ makes bottom 0.}$$

$$= \lim_{x \rightarrow 5} \frac{(x-5)(x+5)}{(x-5)}$$

$$= \lim_{x \rightarrow 5} (x+5) = 5+5 = 10$$

Notice 5 is *not* in the domain. Direct substitution will yield 0/0.

expl 6: Find the limit. If it does *not* exist, say so.

$$\lim_{x \rightarrow -3} \left(\frac{2x^2 - x - 21}{9 - x^2} \right) \leftarrow "-3 \text{ is not in the domain of the rational func"}$$

$$= \lim_{x \rightarrow -3} \frac{(2x-7)(x+3)}{(3-x)(3+x)}$$

$$= \lim_{x \rightarrow -3} \frac{2x-7}{3-x} = \frac{2(-3)-7}{3-(-3)} = \frac{-13}{6}$$

expl 7: Find the limit. If it does *not* exist, say so.

$$\lim_{x \rightarrow 2} \left(\frac{x^2 + 3x - 10}{x^2 - 4x + 4} \right)$$

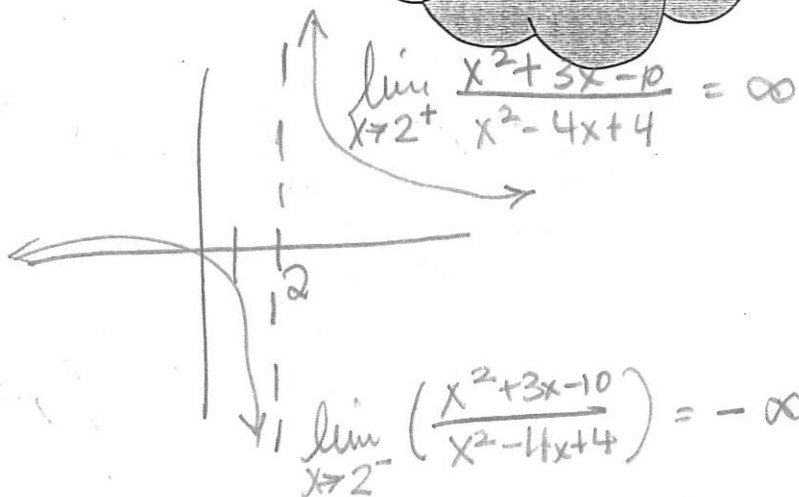
$$= \lim_{x \rightarrow 2} \frac{(x+5)(\cancel{x-2})}{(x-2)\cancel{(x-2)}}$$

$$= \lim_{x \rightarrow 2} \frac{x+5}{x-2}$$

Wait! $x=2$ still makes bottom 0.

Sometimes algebra will *not* help.

Revert to the graphical method. Graph the original function on the **Standard window**.



So, $\lim_{x \rightarrow 2} \frac{x^2 + 3x - 10}{x^2 - 4x + 4}$ d.n.e.

Indeterminate Forms:

When direct substitution gets you 0/0 (or ∞/∞), this is called an indeterminate form. Getting 0/0 means there are common factors on top and bottom of the rational function as we saw earlier. Algebra can help but not always. Try graphical or numerical methods when this happens.

Limits at Infinity:

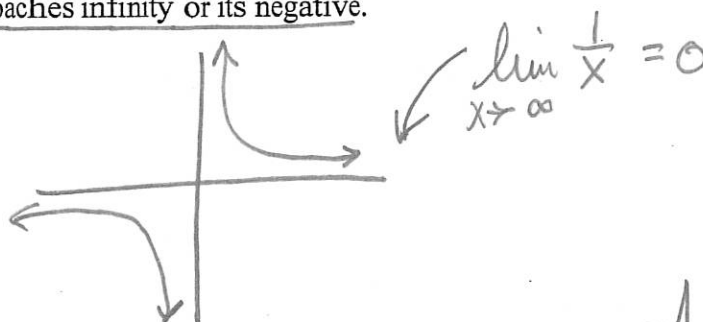
Recall these are limits that we find as x approaches infinity or its negative.

Draw the graph of $y = 1/x$.

Looking at your graph, we see that

$$\lim_{x \rightarrow \pm\infty} \left(\frac{1}{x} \right) = 0.$$

This would be true for the graphs of $y = 2/x, y = 3/x, y = 4/x$, etc.



A little investigating would also convince you that is true of the graphs of

$$y = \frac{1}{x^2}, y = \frac{1}{x^3}, y = \frac{1}{x^4}, \text{ etc. too. We will write this as a general rule: } \lim_{x \rightarrow \pm\infty} \left(\frac{k}{x^n} \right) = 0 \text{ where}$$

k is a real, non-zero number and n is a positive real number. (This differs slightly from our book.)

expl 8: Find the limit. If it does *not* exist, say so.

$$\lim_{x \rightarrow \infty} \left(\frac{2x^2 + 6x - 1}{x^3 + x + 7} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{(2x^2 + 6x - 1) \cdot \frac{1}{x^3}}{(x^3 + x + 7) \cdot \frac{1}{x^3}}$$

$$= \lim_{x \rightarrow \infty} \frac{\frac{2}{x} + \frac{6}{x^2} - \frac{1}{x^3}}{1 + \frac{1}{x^2} + \frac{7}{x^3}}$$

$$= \lim_{x \rightarrow \infty} \frac{0}{1} = \lim_{x \rightarrow \infty} 0 = 0$$

This uses a cool trick. We'll multiply all terms by "1 over x raised to the highest exponent present".

We will then use multiple properties without naming them as such.

L3 & L5

L1

You might, in fact, recall **horizontal asymptotes for rational functions** from algebra. Those can help us figure out these limits too. Here is how the book puts it.

The limit properties can be used to show that if $f(x) = \frac{ax^m + \dots}{bx^n + \dots}$, then

① $\lim_{x \rightarrow \pm\infty} f(x) = \frac{a}{b}$ if $m = n$

② $\lim_{x \rightarrow \pm\infty} f(x) = 0$ if $m < n$

③ and $\lim_{x \rightarrow \pm\infty} f(x)$ does not exist if $m > n$.

(oblique or slant asymptotes)

Which rule here would have given us the limit needed for the previous example? Use them in the future as you like.

Part 2 (if $m < n$)

degree on top = m
degree on bottom = n
(H.A. is found by dividing lead coefficients.)

Precursor for Finding Derivatives:

Here, we see that we need to find the limit “as h approaches some number” of an expression involving both h and x . How do we treat x ?

Well, we treat it as a constant, as you would if it were 2 or 5 or 8. Let’s see.

expl 9: Find the limit. If it does *not* exist, say so.

$$\lim_{h \rightarrow 0} (4x^2 + 5xh + h^2) \stackrel{L3}{=} \lim_{h \rightarrow 0} (4x^2) + \lim_{h \rightarrow 0} (5xh) + \lim_{h \rightarrow 0} (h^2)$$

$\downarrow L1$
 $\downarrow L6$
 $5x \cdot \lim_{h \rightarrow 0} h$

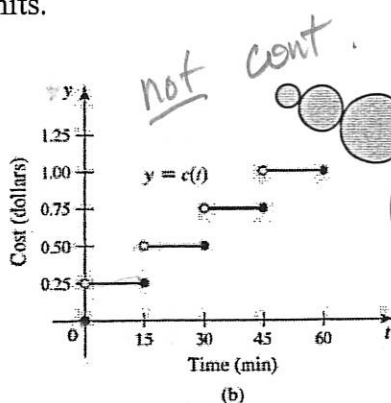
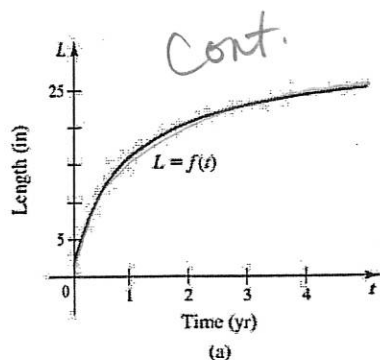
$$= 4x^2 + 5x \cdot 0 + 0$$

$$= 4x^2$$

Can you trace a function’s graph from the far left to the far right without lifting the pencil?

Continuity:

The informal definition of a continuous function appears above. You might have seen that in algebra class. Consider these pictures here to better understand it. We will then move on to a more rigorous definition that uses limits.



The left graph is continuous. Trace it from left to right completely. The right graph is *not* continuous.

Figure 2.42

P if and only if Q

$\rightarrow$ If P , then Q .
 $\rightarrow$ If Q , then P .

Definition: Continuity at a Point: A function is said to be continuous at $x = a$ if and only if $\lim_{x \rightarrow a} f(x) = f(a)$.

More specifically, this means three things.

1.) $f(a)$ is defined. This means that a is in the domain of f .

2.) $\lim_{x \rightarrow a} f(x)$ exists

3.) $\lim_{x \rightarrow a} f(x) = f(a)$

Do you recall how we use right and left-sided limits to find a limit?

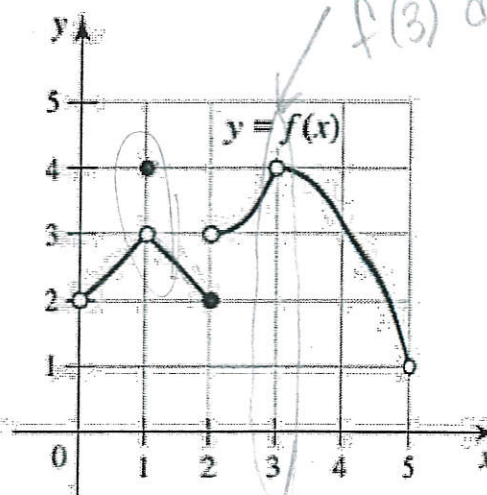
Definition: Point of discontinuity: An x -value where the graph of the function is *not* continuous. You could *not* trace the function without lifting your pencil at this x -value.

expl 10: Find the points of discontinuity in the interval $(0, 5)$. For each point of discontinuity, state which of the three criteria from above is *not* met.

Notice how the interval $(0, 5)$ does *not* include the endpoints.

Interval notation $(0, 5)$

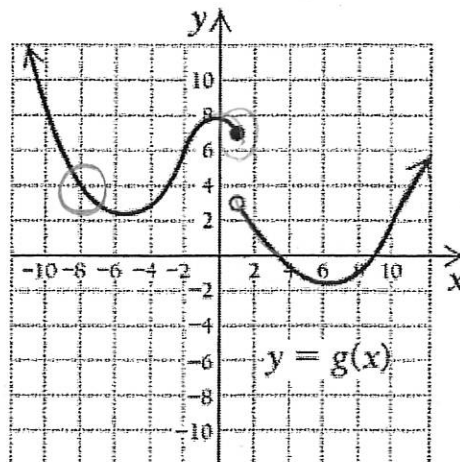
means not to include endpoints as opposed to $[0, 5]$.



Point of Discontinuity (x-value)	Which criteria is <i>not</i> met?
$x = 1$	3) $\lim_{x \rightarrow 1} f(x) \neq f(1)$: $f(1) = 4$, $\lim_{x \rightarrow 1} f(x) = 3$
$x = 2$	2) $\lim_{x \rightarrow 2} f(x)$ dne $\lim_{x \rightarrow 2^-} f \neq \lim_{x \rightarrow 2^+} f$
$x = 3$	1) $f(3)$ is not defined (or does not exist)

To say a function is continuous over an interval means there are *no* points of discontinuity in the interval.

expl 11: Consider the graph of $g(x)$ shown to the right. Take note of the graph's scale. Answer these questions.



a.) Find $\lim_{x \rightarrow -8} g(x)$. $= 4$

b.) Find $g(-8)$. $= 4$

c.) Is g continuous at $x = -8$? yes

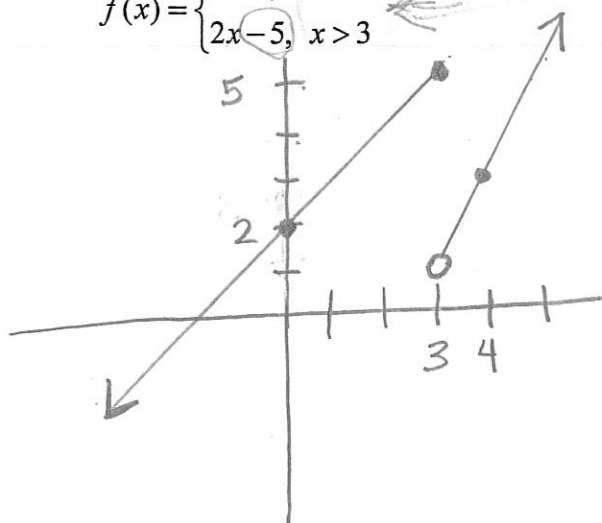
d.) Find $\lim_{x \rightarrow 1} g(x)$. dne

e.) Find $g(1)$. $= 7$

f.) Is g continuous at $x = 1$? No

expl 12: Is the function below continuous at $x = 3$? Explain.

$$f(x) = \begin{cases} x+2, & x \leq 3 \\ 2x-5, & x > 3 \end{cases}$$



A quick graph may help.

No - the func is not cont. at $x = 3$.

I have to lift my pencil to trace the graph at $x = 3$.

OR We see $\lim_{x \rightarrow 3} f(x)$ dne.

expl 13: Find the value of k such that this function would be continuous at $x = 3$.

$$f(x) = \begin{cases} x+2, & x \leq 3 \\ 2x+k, & x > 3 \end{cases}$$

$\Rightarrow$ We need $f(3)$ to be equal to $2x+k$ if we were to substitute $x=3$.

$$x+2 = 2x+k$$

$$3+2 = 2 \cdot 3 + k$$

$$5 = 6 + k$$

$$\underline{-1 = k}$$

$$\text{So, } f(x) = \begin{cases} x+2, & x \leq 3 \\ 2x-1, & x > 3 \end{cases}$$

will be cont at $x = 3$.