

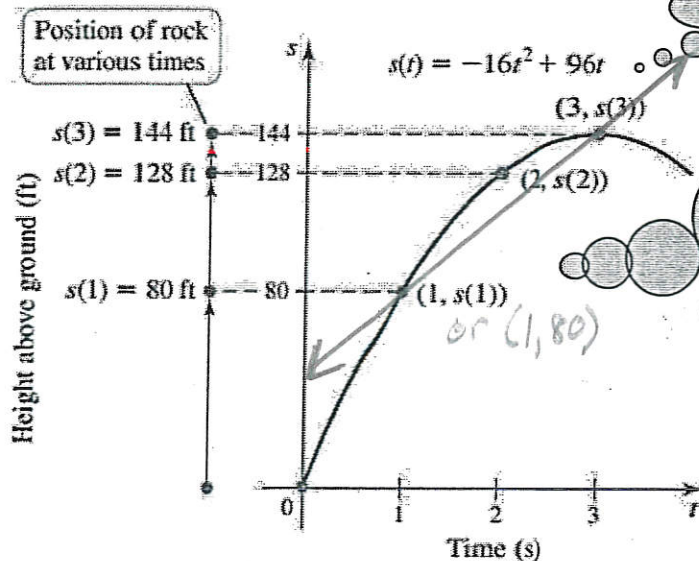
You have seen this before as the slope of a line. Here, we apply the notion to nonlinear functions using secant lines.

Consider a car that travels 240 miles in 4 hours. We say its **average velocity** is $240 \text{ miles} / 4 \text{ hours} = 60 \text{ miles per hour}$.

But that's only an *average*. Sometimes it goes slower and sometimes faster than 60 mph. Its *velocity at any instant* is its **instantaneous velocity**. We'll look at that later.

Launch a rock upward and let it fall to earth. Let's explore its velocity more closely.

We'll use $s(t) = -16t^2 + 96t$ for the rock's height in feet (also called its **position**) $s(t)$ at time t seconds. Here's a partial graph.



This is *not* the rock's path.

Note how y -values or heights are determined.

Use a straight edge to draw a **secant line** through the points for $t=1$ and $t=3$.

A **secant line** hits a graph at two points. A **tangent line** hits only one point.

expl 1: Find the **average velocity** of the rock by finding the slope of the line through these two points $(1, 80)$ and $(3, 144)$. Include units. This is what is known as the **average rate of change** of the height of the rock.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{144 - 80}{3 - 1}$$

$$= \frac{64}{2}$$

$$\rightarrow m = 32 \text{ ft/sec}$$

Definition: Average Rate of Change:

The **average rate of change** in f on the interval $[x_1, x_2]$ is the slope of the secant line between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. We'll assume $x_1 \neq x_2$. Do you see why the formula requires that?

$$* m_{\text{sec}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Recall the formula for slope. Here, we see it used with function notation.

The slope of a tangent line, which is the **instantaneous rate of change** of the function, takes on different values as you traverse the graph. We will see that in the next section. To get there, we usually use another form for this average rate of change.

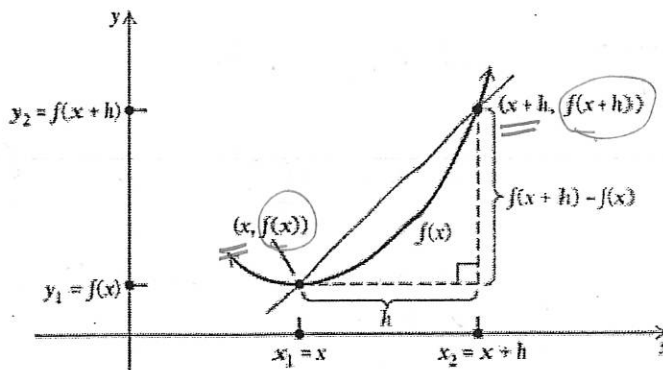
Alternative Definition Using the Difference Quotient:

Imagine two points on a graph. One is a little bit to the right of the other.

We use h to denote "a little bit". We assume h is not 0.

That means that our two points are $(x, f(x))$ and $(x+h, f(x+h))$.

We find the slope between these two points connected by the secant line from before. Do that now and simplify.



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x+h - x} = \frac{f(x+h) - f(x)}{h} \quad \text{Difference Quotient}$$

This formula is called the **difference quotient**. As before, this might be called the **average rate of change** of f (with respect to x) and the limit (as h approaches 0) of this quotient will be the function's **instantaneous rate of change** (provided the limit exists). Again, that part comes later.

ex1 2: Use your formula to find the difference quotient for the function $f(x) = 3x^2$. Simplify.

$$\begin{aligned} \text{Diff Quot} &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{3x^2 + 6hx + 3h^2 - 3x^2}{h} \\ &= \frac{6hx + 3h^2}{h} \end{aligned}$$

$$= \frac{h(6x + 3h)}{h}$$

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$$\text{Diff Quot} = 6x + 3h$$

Cause if $x_1 = x_2$, we'd be dividing by 0.

$$\begin{aligned} f(x+h) &= 3(x+h)^2 \\ &= 3(x+h)(x+h) \\ &= 3(x^2 + 2hx + h^2) \\ f(x+h) &= 3x^2 + 6hx + 3h^2 \end{aligned}$$

Difference Quotient:

The average rate of change of $f(x)$ with respect to x is also called the difference quotient. It is given by

$$\frac{f(x+h) - f(x)}{h}, \text{ where } h \neq 0.$$

The difference quotient is equal to the slope of the secant line passing through $(x, f(x))$ and $(x+h, f(x+h))$.

expl 3a: Use your formula to find the difference quotient for the function $f(x) = x^3$. Simplify.

$$\begin{aligned} \text{Diff Quot} &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \end{aligned}$$

$$= \frac{\cancel{x}(3x^2 + 3hx + h^2)}{\cancel{x}}$$

$$DQ = 3x^2 + 3hx + h^2$$

$$\begin{aligned} f(x+h) &= (x+h)^3 \\ &= (x+h)(x+h)(x+h) \\ &= (x^2 + 2hx + h^2)(x+h) \\ &= x^3 + 2hx^2 + h^2x + hx^2 + 2h^2x + h^3 \\ &= x^3 + 3hx^2 + 3h^2x + h^3 \end{aligned}$$

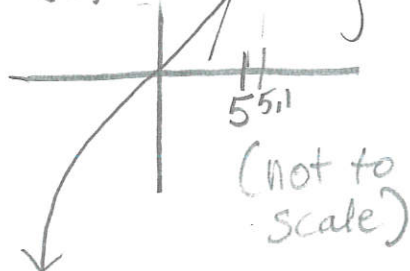
$$f(x+h) = x^3 + 3hx^2 + 3h^2x + h^3$$

expl 3b: Use the formula from above to fill in the table to the right. Draw a quick graph of the function with the two points $(5, f(5))$ and $(5.1, f(5.1))$ labeled.

x	h	$\frac{f(x+h) - f(x)}{h}$
5	1	91
5	0.1	76.51
5	0.01	75.1501

$$DQ: 3x^2 + 3hx + h^2$$

$(5.1, 132.651)$
 $(5, 125)$



The slope of the secant line here is 76.51.

As h gets smaller, the two points become closer and closer.

Means

Year = 2010 $\rightarrow t = 10$

expl 4: The condor population in the Grand Canyon in Arizona can be approximated by

$P(t) = 2.8t^{1.87}$ where t is the number of years since 2000. Find $\frac{P(13) - P(10)}{13 - 10}$ and explain what it represents.

$$\frac{P(13) - P(10)}{13 - 10} \approx 43.82$$

Condors/year

$$P(13) = 2.8(13)^{1.87}$$

$$P(10) = 2.8(10)^{1.87}$$

This means that the average rate of change in the condor population from 2010 to 2013 is 43.82 condors per year. On average, the population is growing by 43.82 Condors per year.

expl 5: Use your formula to find the difference quotient for the function $f(x) = \frac{1}{x^2}$. Simplify.

$$DQ = \frac{f(x+h) - f(x)}{h}$$

$$= \frac{-2hx - h^2}{x^2(x+h)^2} \quad \text{(from work on pg 5)}$$

$$= \frac{-2hx - h^2}{x^2(x+h)^2} \cdot \frac{1}{h}$$

$$= \frac{\cancel{h}(-2x - h) \cdot 1}{x^2(x+h)^2 \cdot \cancel{h}}$$

$$DQ = \frac{-2x - h}{x^2(x+h)^2}$$

This one's a stinker.
Calculus is hard but
algebra's harder!

$$f(x) = \frac{1}{x^2}$$

$$f(x+h) = \frac{1}{(x+h)^2}$$

$$f(x+h) - f(x) = \frac{1}{(x+h)^2} - \frac{1}{x^2}$$

$$\text{LCD} = x^2(x+h)^2$$

$$= \frac{1 \cdot x^2}{(x+h)^2 \cdot x^2} - \frac{1 \cdot (x+h)^2}{x^2(x+h)^2}$$

$$= \frac{x^2 - (x+h)^2}{x^2(x+h)^2} \rightarrow \dots$$

(more room for work)

$$\frac{x^2 - (x+h)^2}{x^2(x+h)^2} = \frac{x^2 - \overbrace{(x+h)(x+h)}^{\text{FOIL}}}{x^2(x+h)^2}$$

$$= \frac{x^2 - (x^2 + 2hx + h^2)}{x^2(x+h)^2}$$

$$= \frac{x^2 - x^2 - 2hx - h^2}{x^2(x+h)^2} = \frac{-2hx - h^2}{x^2(x+h)^2}$$

