

We have explored average rates of change. Now, we use limits to find instantaneous rates of change.

Business Calculus

Class Notes

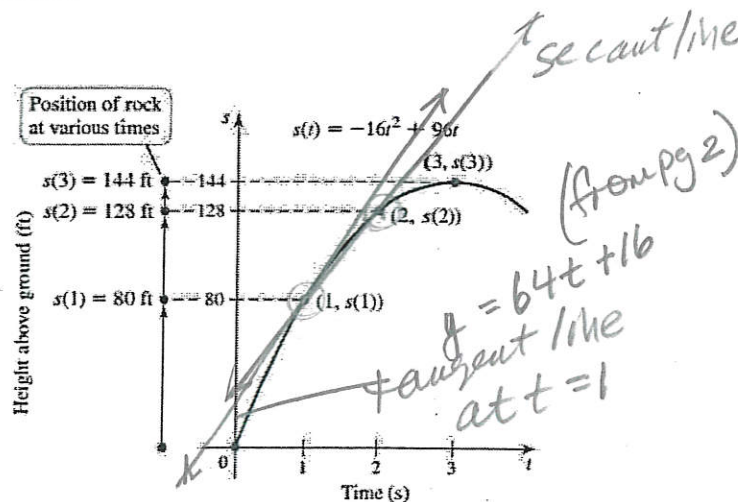
Section 1.4: Differentiation Using Limits of Difference Quotients

The slope of a tangent line, which is the **instantaneous rate of change** of the function, takes on different values as you traverse the graph. Recall the graph below we were exploring when we first thought about average rates of change.

Recall this is the partial graph for the function $s(t) = -16t^2 + 96t$ where $s(t)$ is the height of a rock t seconds after it has been thrown straight up.

★ Draw in the secant line through the t -values 1 and 2.

Now, imagine this second point traveling along the function's curve getting closer and closer to the first point at $t = 1$.



The slope of this secant line is the difference quotient $\frac{f(x+h) - f(x)}{h}$ from the previous section where we would use 1 for x and 1 for h (because that's the difference between our x -values). How do we notate that this second point is getting closer and closer to the first point? Or rather, the value of h is approaching zero?

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Do you see how the secant line coincides with the tangent line when h is 0?

When we find this secant line's slope, we find the average rate of change of the rock's height.

★ When we find this slope as we imagine h approaching 0, we find the **instantaneous rate of change**. Good thing we just studied limits!

which is the slope of the tangent line.

$$\begin{aligned}
 s(t+h) &= -16(t+h)^2 + 96(t+h) \\
 &= -16(t^2 + 2ht + h^2) + 96t + 96h \\
 &= -16t^2 - 32ht - 16h^2 + 96t + 96h
 \end{aligned}$$

expl 1: The rock's position (height, in feet) above the ground t seconds after release is given by $s(t) = -16t^2 + 96t$. Find the instantaneous rate of change (or velocity) at t seconds. Then, use your result to find the instantaneous rate of change at 1 second.

$$v_{inst} = \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$$

$$\begin{aligned}
 &\frac{s(t+h) - s(t)}{h} \\
 &= \frac{-16t^2 - 32ht - 16h^2 + 96t + 96h - (-16t^2 + 96t)}{h}
 \end{aligned}$$

$$= \frac{-\cancel{16t^2} - 32ht - 16h^2 + \cancel{96t} + 96h + \cancel{16t^2} - \cancel{96t}}{h}$$

$$= \frac{-32t - 16h + 96}{1}$$

$$\text{So, } v_{inst} = \lim_{h \rightarrow 0} (-32t - 16h + 96)$$

$$v_{inst} = -32t + 96$$

Interpretation:

Our result means that the projectile's height is changing at the rate of 64 feet per second when the projectile has been in the air for 1 second. We define the rate of change for a position function to be the object's velocity.

For now, concentrate again on this point (1, 80) and the fact that we know the slope of the tangent line here is 64. Use good ol' algebra to find the equation of this tangent line that goes through this point. Use *either* starting formula below.

Point-slope Formula

$$y - y_1 = m(x - x_1)$$

$$y - 80 = 64(x - 1)$$

$$y - 80 = 64x - 64$$

$$y = 64x + 16$$

OR

Slope-intercept Formula

$$y = mx + b$$

$$80 = 64(1) + b$$

$$16 = b$$



$$y = 64x + 16$$

For the first part, leave t in place. We'll substitute 1 for t for the second part.

→ The instantaneous rate of change (velocity) at $t=1$ is $v_{inst} = -32(1) + 96$

$$= 64 \text{ feet/Second}$$

The units come from the formula for slope. We use units for $s(t)$ divided by units of t .

Go back to the first page to graph and label the tangent line.

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Our independent variable was really t , So eqn is really $y = 64t + 16$.

We start off with a projectile example because it's nice to imagine the rock and how its position varies over time. However, we can do this sort of thing with any function.

Maybe we have a function that describes a deer population, or a business' revenue, or the weight of a child. For any function, we can find its average rate of change (slope of secant line) and then use limits to find its instantaneous rate of change (slope of tangent line).

By the way, for any point on a function's graph, the tangent line is unique (there's only one) and hence the slope is unique. That is why we will call this process a function. Do you remember what makes a relationship a function?

one input yields exactly one output

This process of finding the slope of the tangent line is called **differentiation**. We are, in fact, finding the **derivative** of our function. So, let's have some formal definitions and formulas.

Definition: Derivative:

For a function f , given by $y = f(x)$, its derivative at x is the function f' , where

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

"f prime"

f'

provided that the limit exists. If $f'(x)$ exists, then we say that f is **differentiable** at x .

The equation of the tangent line (at $x = a$) will always be $y - f(a) = f'(a)(x - a)$.

ex1 2: For the following function, complete the following.

- Find the slope of the tangent line to f at $P = (1, -2)$.
- Find the equation of the tangent line.

$$f(x) = -3x^2 + 1$$

$$f'(1)$$

Find $f'(1)$. Write small.

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-6x - 3h}{1}$$

$$f'(x) = \lim_{h \rightarrow 0} (-6x - 3h)$$

$$\text{So, } f'(x) = -6x$$

$$f'(1) = -6(1) = -6$$

$$y - f(a) = f'(a)(x - a)$$

$$y - -2 = -6(x - 1) \Rightarrow y = -6x + 4$$

$$f(x) = -3x^2 + 1$$

$$f(x+h) = -3(x+h)^2 + 1$$

$$= -3(x+h)(x+h) + 1$$

$$= -3(x^2 + 2hx + h^2) + 1$$

$$f(x+h) = -3x^2 - 6hx - 3h^2 + 1$$

$$f(x+h) - f(x)$$

$$= -3x^2 - 6hx - 3h^2 + 1 - (-3x^2 + 1)$$

$$= -3x^2 - 6hx - 3h^2 + 1 + 3x^2 - 1$$

$$= -6hx - 3h^2$$

A Logic Sidebar:

Definition: Conditional Statement: An "If, then" statement like many of our theorems have been. Often, we write a generic conditional as "If p , then q ."

Definition: Contrapositive Statement: The contrapositive of the above statement is "If *not* q , then *not* p ."

If a conditional statement is true, then its contrapositive is always true as well. A standard example of this is the following.

Assume "If it rains, then I will drive you home." is true.

Notice that "If I *don't* drive you home, then it did *not* rain." *must* also be true. The reasoning being if it rains, then you were driven home. Since I did *not* drive you home, then it could *not* have rained.

Connection Between Continuity and Differentiability:

We have this theorem and its contrapositive.

Theorem: If f is differentiable at a , then f is continuous at a .

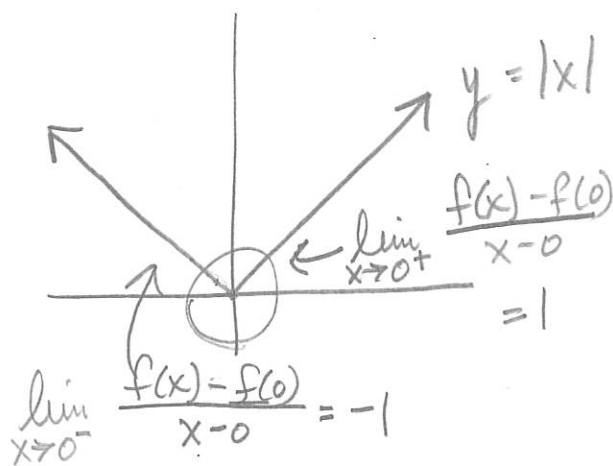
Contrapositive: If f is *not* continuous at a , then f is *not* differentiable at a .

Both are true statements. The book gives us the second.

A Common Misconception: The logic flow of a theorem is important. The statement "If f is continuous at a , then f is differentiable at a ." is NOT true.

Here is an example that shows this to be false.

Draw a quick graph of $y = |x|$. Notice it is continuous over its entire domain $(-\infty, \infty)$. But is it differentiable for all x values?



We know $f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$ or, rather,

consider the point $(0, 0)$ and another point very close by.

Move this point closer and closer to $(0, 0)$ to find this limit from the left? From the right? Do they match? What does that mean about

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} ?$$

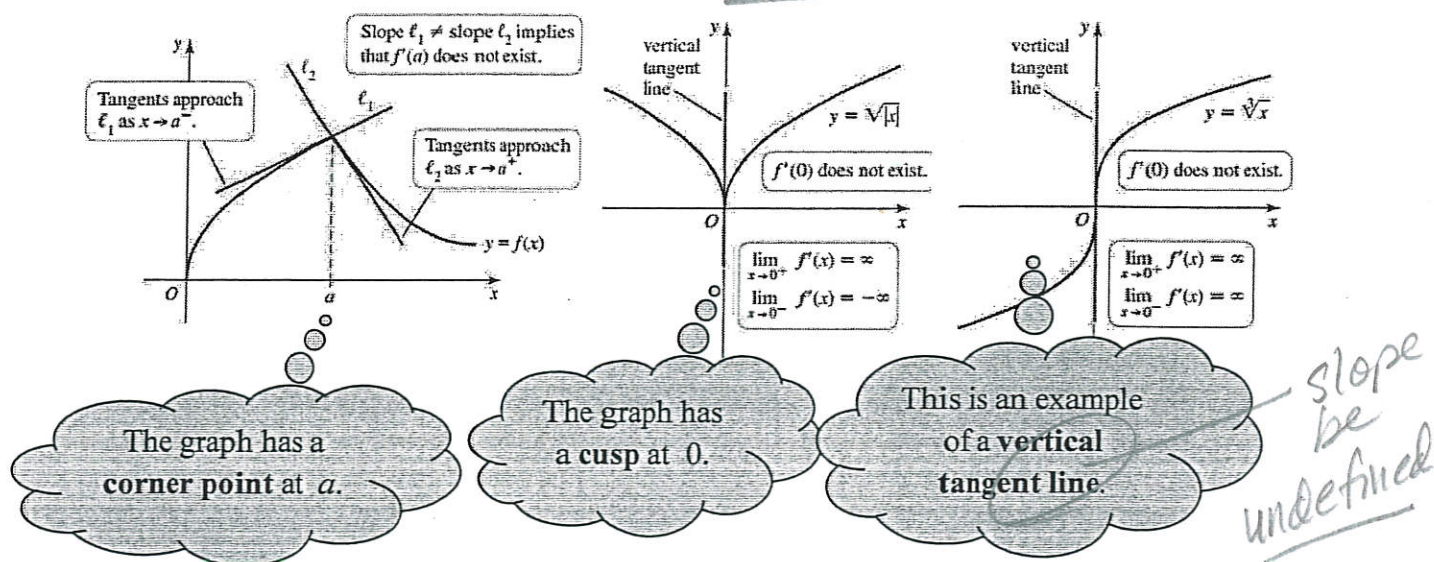
d.n.e. because

$$\lim_{x \rightarrow 0^-} \neq \lim_{x \rightarrow 0^+}$$

So, even though this function is continuous over $(-\infty, \infty)$, it is *not* differentiable at $x = 0$.

More Examples:

Here are three scenarios that can cause a function to not be differentiable at a point.



Another book provides us with this nice summary which you might find useful.

When Is a Function Not Differentiable at a Point?

A function f is not differentiable at a if at least one of the following conditions holds:

- f is not continuous at a
- f has a corner at a
- f has a vertical tangent at a

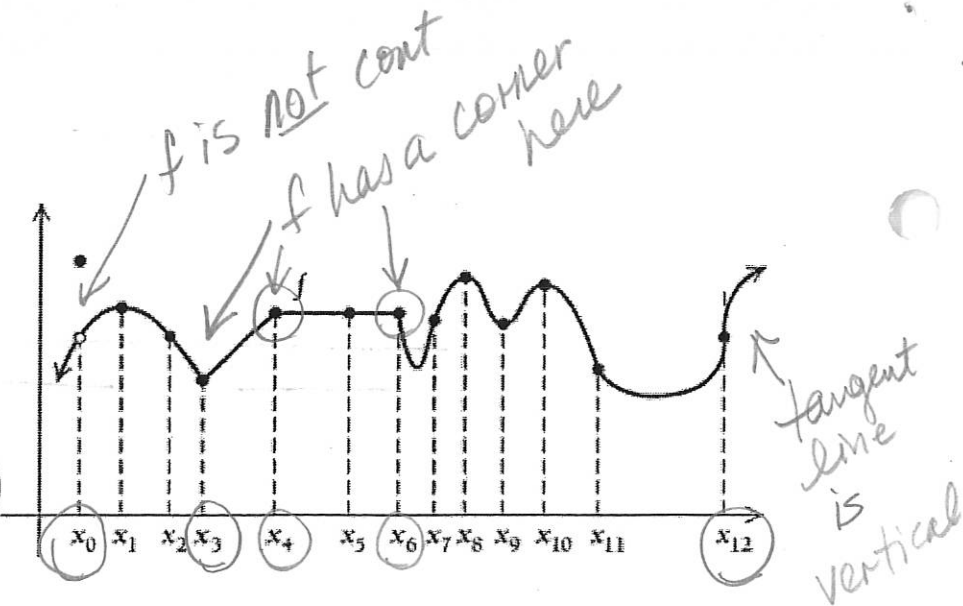
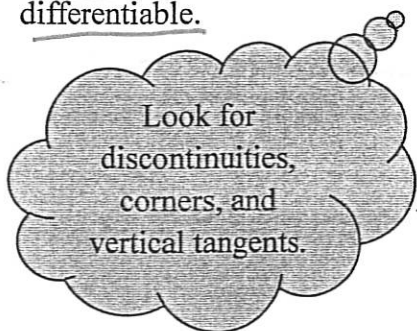
We will wrap with this distinction.

Continuity (for a function f) requires $\lim_{x \rightarrow a} f(x) = f(a)$ or written in another way,

$$\lim_{x \rightarrow a} (f(x) - f(a)) = 0.$$

Differentiability requires more. It requires that $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists. (This uses an alternative formula for the derivative based on the fact that it is the limit, as the two points get closer, of the slope of the secant line with which we started.)

expl 3: For the graph to the right, list all x-values for which the function is not differentiable.



expl 4: Draw a graph of a function that meets *all* of the qualities below.

The domain of the function is $[0, 7]$.

The graph has a horizontal tangent line at $x = 5$. $\leftarrow \text{slope} =$

The graph is continuous at all values except $x = 2$.

The function is *not* differentiable (but is continuous) at $x = 4$.

