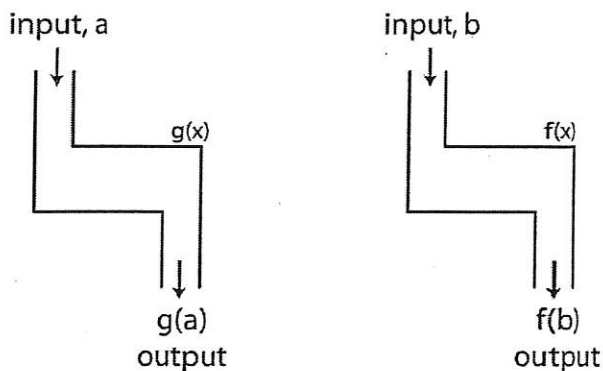


Business Calculus
Class Notes
Section 1.7: The Chain Rule

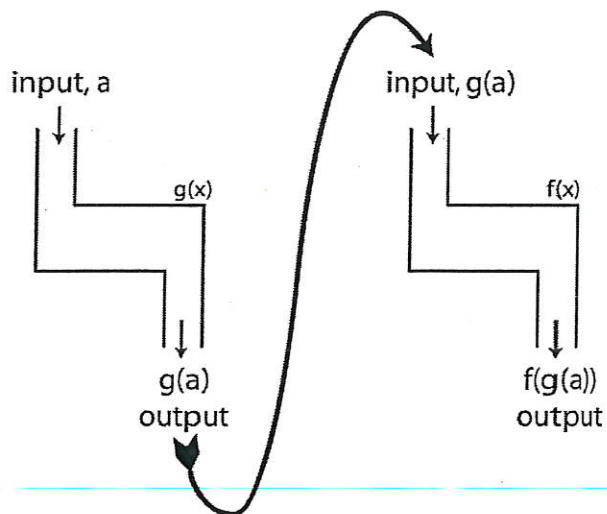
Many functions are the composition of two functions whose derivative rules we know.

A Review of Composition:

It will help if you think of functions as machines that take an input and produce an output. Consider two functions, $f(x)$ and $g(x)$, pictured below.



Let's suppose we apply the function g to a number, **and then** put that output into the function f . That is the idea of composition and is pictured below.



Notice how the final output is denoted.

If plain old function notation is hard for you, review it before attempting this section.

Composition Notation: The following notations are equivalent.

$$(f \circ g)(x) = f \circ g = f(g(x))$$



Use the outputs of g as the inputs of f .

Pronounced
"f of g of x"

expl 1: The function $h(x) = \sqrt{4x^2 - 3}$ is the composition of two functions. What are those functions? Call them $g(x)$ and $f(x)$ for the inside (happens first) and the outside function. You can check your answer by finding $f(g(x))$ and seeing if it really is equal to the original $h(x)$.

$$g(x) = 4x^2 - 3$$

$$f(x) = \sqrt{x} \quad \rightarrow \quad f(g(x)) = f(4x^2 - 3) = \sqrt{4x^2 - 3} \text{ which is } h(x) \checkmark$$

We know that $\frac{d}{dx} \sqrt{x} = \frac{1}{2} x^{-1/2}$ but we cannot simply say that $\frac{d}{dx} \sqrt{4x^2 - 3} = \frac{1}{2} (4x^2 - 3)^{-1/2}$

because the inside function is not just x . In fact, as we just saw, this $4x^2 - 3$ can be thought of as a function unto itself. And that's the key. We need to start seeing some functions as the composition of two functions whose derivative rules we know.

We have this theorem from our book.

The Chain Rule

The derivative of the composition $f \circ g$ is given by

$$\frac{d}{dx} [(f \circ g)(x)] = \frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

expl 2: Use the Chain Rule to find the derivative of $h(x) = \sqrt{4x^2 - 3}$.

$$h(x) = f(g(x)) \quad f(x) = \sqrt{x}, \quad g(x) = 4x^2 - 3$$

$$h'(x) = \frac{1}{2} (4x^2 - 3)^{-1/2} \cdot (8x - 0)$$

$$h'(x) = 4x (4x^2 - 3)^{-1/2}$$

We multiply by the derivative of the inside function to complete our answer.

expl 3: Find $\frac{d}{dx}((2x+6)^3)$.

$$h(x) = (2x+6)^3$$

$$(f \circ g)(x) = (2x+6)^3 \quad \text{Let } g(x) = \underline{2x+6}$$

$$f(x) = \underline{x^3}$$

$$= f'(g(x)) \cdot g'(x)$$

$$= 3(2x+6)^2 \cdot 2 = \boxed{6(2x+6)^2}$$

$$f'(x) = 3x^2$$

$$g'(x) = 2 \cdot 1x^{1-1} + 0 = 2$$

The situation we see in the previous example, some function raised to a power, happens a lot. So often, we are given a separate rule. It really is just the Chain Rule but sometimes it's nice to use.

The Extended Power Rule

Suppose that $g(x)$ is a differentiable function of x . Then, for any real number k ,

Specific instance where $f(x) = x^k$

$$g(x) = 2x+6 \quad k=3$$

$$\frac{d}{dx} [g(x)]^k = k[g(x)]^{k-1} \cdot \frac{d}{dx} g(x).$$

expl 4: Find $\frac{d}{dx}(\sqrt{x^2+1})$. $= \frac{d}{dx}((x^2+1)^{1/2})$

$$= \frac{1}{2} (x^2+1)^{-1/2} \cdot 2x$$

$$= \boxed{x(x^2+1)^{-1/2}}$$

$$g(x) = x^2+1$$

$$k = 1/2$$

$$\frac{d}{dx}(g(x)) = \frac{d}{dx}(x^2+1) = 2x$$

Product Rule, anyone?

expl 5: Find y' for $y = x(x+2)^{1/3}$.

$$y' = \overset{f'}{\downarrow} 1 \cdot \overset{g}{\downarrow} (x+2)^{1/3} + \overset{\text{plus}}{\downarrow} \overset{f}{\downarrow} x \cdot \overset{g'}{\downarrow} \frac{1}{3} (x+2)^{-2/3}$$

$$\boxed{y' = (x+2)^{1/3} + \frac{x}{3(x+2)^{2/3}}}$$

OR

$$\boxed{y' = (x+2)^{1/3} + \frac{1}{3} x (x+2)^{-2/3}}$$

$$\frac{d}{dx}(f \cdot g) = \underline{f'g} + \underline{fg'}$$

$$\text{Let } f(x) = x$$

$$g(x) = (x+2)^{1/3}$$

$$g'(x) = \frac{1}{3} (x+2)^{-2/3} \cdot 1$$

- The derivative is the slope of the tangent line.

Recall - $y' = 6(2x+6)^2$ from exl 3

exl 6: Find the equation of the tangent line to the graph of $y = (2x+6)^3$ at the point $(2, 1000)$.

Find slope of tangent line at $x=2$.

$$y' = 6(2 \cdot 2 + 6)^2 = 6(10)^2 = 600$$

Now find the eqn of this tangent line:

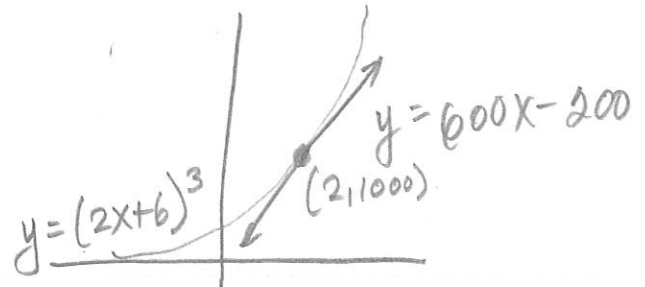
$$y - y_1 = m(x - x_1) \leftarrow \text{(Point-slope form)}$$

$$y - 1000 = 600(x - 2)$$

$$y - 1000 = 600x - 1200$$

$$y = 600x - 200$$

$$[-5, 5] \times [-10, 2000]$$



Alternative Notation:

THEOREM 3.12 The Chain Rule $y = f(g(x))$
 Suppose $y = f(u)$ is differentiable at $u = g(x)$ and $u = g(x)$ is differentiable at x .
 The composite function $y = f(g(x))$ is differentiable at x , and its derivative can be expressed

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad (1)$$

der of y with respect to u
 der of $y = f(g(x))$ with respect to x
 der of u with respect to x

This comes from another book. It can be helpful to see the outside function as y (which is a function of u) and the inside function as u (which is a function of x). This notation is kind of handy as it feels a little like we're "cancelling" du on the right side to get dy/dx on the left.

The next example describes two functions. One defines a function E in terms of U . The second defines a function U in terms of x . We will explore how E changes with respect to x .



expl 7: A woman's US shoe size is determined by the length of her foot. In fact, $U(x) = 3x - 21$ where $U(x)$ is her US shoe size and x is the length of her foot, in inches. Further, we know that a woman's European shoe size $E(U)$ is related to her US shoe size U by $E(U) = 1.35U + 28.25$. Answer the following to find the rate of change of her European shoe size with respect to the length of her foot.

a.) Find and interpret $\frac{dU}{dx} = 3$ The rate of change of shoe size with respect to the length of her foot.

b.) Find and interpret $\frac{dE}{dU} = 1.35$ The rate of change of European shoe size with respect to her US shoe size.

c.) Use the fact that $\frac{dE}{dx} = \frac{dE}{dU} \cdot \frac{dU}{dx}$ to find and interpret $\frac{dE}{dx}$.

$$= 1.35 \cdot 3$$

$$= \underline{\underline{4.05}}$$

As x increases by 1 inch, her European shoe size increases by 4.05.

expl 8: Find $\frac{d}{dx} \left(\frac{3}{(x^3 + 4x)^2} \right)$.

$$= \frac{d}{dx} \left(3(x^3 + 4x)^{-2} \right) \quad f(x) = x^{-2}$$

$$= 3(-2)(x^3 + 4x)^{-3} (3x^2 + 4)$$

$$= \frac{-6(3x^2 + 4)}{(x^3 + 4x)^3}$$

OR $-6(3x^2 + 4)(x^3 + 4x)^{-3}$

OR $(-18x^2 - 24)(x^3 + 4x)^{-3}$

$$\frac{1}{a^n} = a^{-n}$$

Sometimes it looks like the quotient rule is needed but it's not. Do you see how?

Use only Chain Rule

