I'm keeping track of all the exponentials I can find. Every time I see one, I log it.

Business Calculus
Class Notes
Section 2.1: Review: Exponential and Logarithmic Functions

We need to talk about the rates of change for a whole class of functions called natural exponential and logarithmic functions. Before we get to their derivatives, we should do a quick review of what these functions do for us. We'll talk about how we manipulate them including how to solve equations involving them. So, please step back in your mind and recall…

Recall: Definition: Exponential Function: — like $y = 2^x$ or $y = \left(\frac{1}{3}\right)^x$

An **exponential function** is of the form $f(x) = a^x$ where a is a positive real number *not* equal to 1. The number a is called the **base** or **growth factor**. Notice the variable x is in the exponent position. This is what makes it an exponential function.

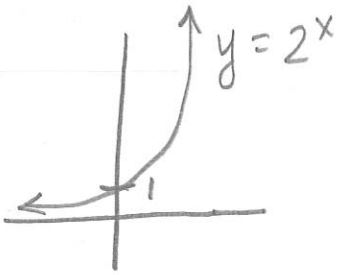
We will also see functions in the form $f(x) = Ca^x$. Here, C is a real number *not* equal to 0 and it is called the **initial value**. Do you see why?

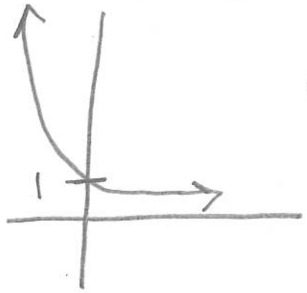
like $y = 4 \cdot 3^x$

$f(0) = C \cdot a^0 = C$

Domain: Since you could raise a positive number to any power and get a real number out, there are *no* real numbers we need to exclude from the domain. So, **the domain of any exponential function is all real numbers.**

Give yourself a quick graph of something like $f(x) = 2^x$. They have a distinctive shape. If you want to make up your own formula, do so but write it down.





Do you remember what happens to the graph when the base is less than 1?

Recall: Definition: Natural Exponential Function:

This is a special exponential function whose base is e, an irrational number (meaning its decimal form does *not* terminate or repeat) approximately equal to 2.71828. The **natural exponential function** is $f(x) = e^x$.



domain: $(-\infty, \infty)$
range: $(0, \infty)$

The Inverse of the Natural Exponential Function:

So far, our function, $f(x) = e^x$, takes an x-value and returns (as a y-value) "e raised to that x-value". Inverses undo each other and that is what we seek. We want to be able to start with some number we know is "e raised to some x-value" and figure out the x-value that generated it.

For reasons that would have been covered in an algebra class, we know the inverse of the natural exponential function is the natural logarithmic function. With a change of variable (because we like to use x for inputs), we define it like this.

We say this as "log, base e, of x".

Recall: Definition: Natural Logarithmic Function:

We define $y = \log_e x$ to be the number y such that $x = e^y$. Because of its connection to the exponential relationship, we say $x > 0$ (this is the domain of the function).

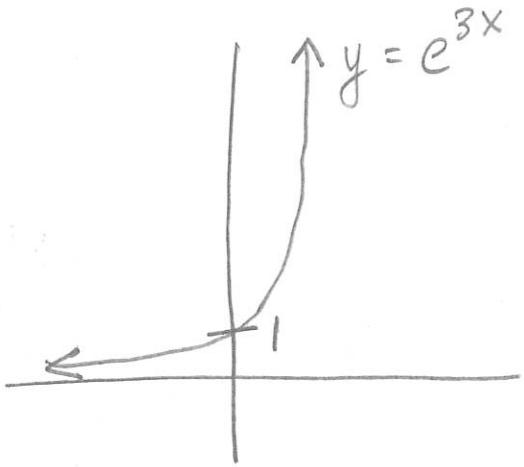
Sometimes written as $y = \log_e(x)$ or $y = \ln(x)$.

A more useful way to define logs is that $\ln x$ is the number to which I raise e to get x.

Notice the base of the logarithmic function is the same as the base of the exponential function from which it came. (The base can be any value of a that is a positive real number *not* equal to 1. Those are simply logarithmic functions (*not* natural). However, here we focus on base e.)

expl 1: Graph the function. Determine its domain, range, y-intercept, and whether the function is increasing or decreasing.

$y = e^{3x}$

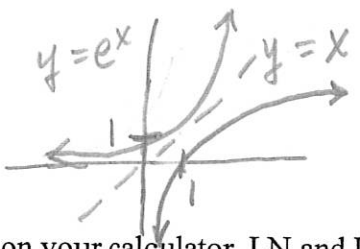


D: $(-\infty, \infty)$

R: $(0, \infty)$

y-int: $(0,1)$ [the point]

The fnc is increasing.



inverse
$x = e^y$ or $y = \ln x$
(domain $x > 0$)

Calculator usage:
You will see two buttons on your calculator, LN and LOG. These are base e and base 10 logs. Keep them straight by noticing the second functions of each button.

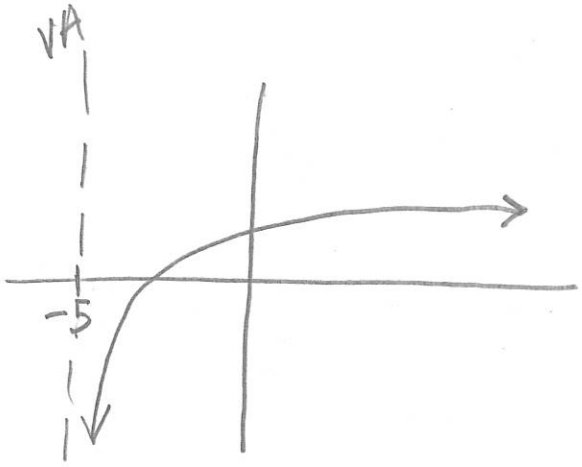
expl 2a: Find this value using the calculator. Round to three decimal places.
ln 80.56 $\approx$ 4.389

OR $\log_e 80.56 \approx 4.389$

expl 2b: The value we got for ln 80.56 completes an exponential equation. Fill in the two missing parts of this equation to make it true. Be sure to check it on your calculator.

$e^{4.389} \approx 80.56$

expl 3: Graph the function. Determine the domain of the function. Do some algebraic work to inform your graph. Draw in the vertical asymptote as a dashed line.
$f(x) = \ln(2x+10)$



Domain: $2x + 10 > 0$
$x > -5$ Domain

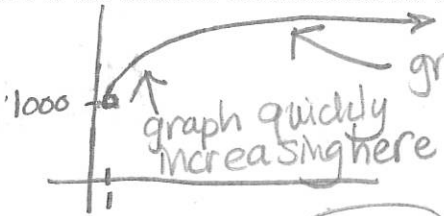
expl 4: A model for advertising response is given by $N(a) = 1000 + 200 \ln a$, $a \geq 1$. Here $N(a)$ is the number of units sold when a thousand dollars is spent on advertising.

What is a?
$a = 5$

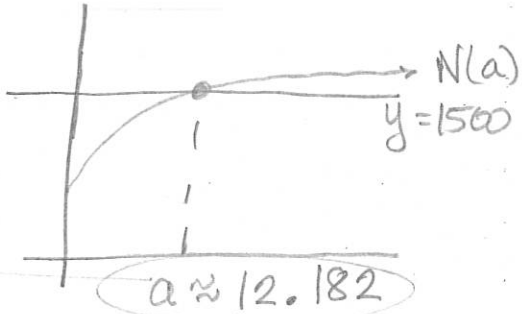
a.) How many units would be sold if they spend $5,000 on advertising?

$N(a) = 1000 + 200 \cdot \ln a$
$N(5) = 1000 + 200 \cdot \ln 5 \approx 1322$ units sold

b.) Graph the function on the window [0, 25] x [0, 2000].
What happens to the number of units sold as a increases?



c.) The company would like to sell 1500 units. How much should be spent on advertising? Solve graphically.



$N(a) = 1000 + 200 \ln a$
$1500 = 1000 + 200 \cdot \ln a$

$a \approx 12.182$

→ To sell 1500 units, we should spend $12,182 on ads.

Review of Exponent Rules: You will use many of these rules as you manipulate expressions.

Product rule: $a^m \cdot a^n = a^{m+n}$

Quotient rule: $\frac{a^m}{a^n} = a^{m-n}$

Power rule: $\left(a^m\right)^n = a^{m \cdot n}$

Power of a product rule: $(a \cdot b)^n = a^n \cdot b^n$

Power of a quotient rule: $\left(\frac{a}{c}\right)^n = \frac{a^n}{c^n}$

Zero exponent rule: $a^0 = 1$ (Here a cannot be 0 because 0^0 is undefined.)

Negative exponent rule: $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$ (if a is non-zero and n is an integer).

Recall: Properties of Natural Logarithms:
Often, these rules are studied for any defined base but here I wrote them with only base e in mind. These rules are only true if the logs are defined. For instance, many rules require that $M > 0$. The first rule requires that $N \neq 0$. Do you see why?

1. $\ln\left(M/N\right) = \ln M - \ln N$
2. $\ln(M \cdot N) = \ln M + \ln N$
3. $\ln M^r = r \cdot \ln M$
4. $\ln e = 1$
5. $\ln 1 = 0$
6. $\ln e^r = r$
7. $e^{\ln M} = M$
8. $\log_b M = \frac{\ln M}{\ln b}$

expl 5: Simplify.

a.) $\ln(4e) = \ln 4 + \ln e = \ln 4 + 1$ (Prop 2; Prop 4)

log$_e$e is the number I raise e to, to get e

$e^? = e$

$e^1 = e$

b.) $\ln e^6 = 6$ (Prop 6)

Review: Algebraically Solving Exponential and Logarithmic Equations:
We will practice our algebra skills by solving equations involving natural exponential and logarithmic functions. Some examples follow with a few with bases other than e thrown in.

Recall: Definition: Exponential Equation: An equation with the variable in the exponent position.

Examples: $5^x = 5^4$, $e^{2t} = 500$, $3^{4x} = 81$, $3^r = 2^{r-1}$, $e^x - 6e^{-x} = 1$

Recall: Definition: Logarithmic (Log) Equation: An equation that has logs of variable expressions.

Examples: $\ln(4w) = 4$, $3\ln(x-1) + \ln 4 = 5$, $\ln(x-1) - \ln(x+2) = 2$

We're going to be asked to solve these equations. Remember we are trying to find the x that makes the equation true. We will need to keep a lot in our minds.

1.) the logarithm properties from earlier
2.) the fact that $x = a^y$ and $y = \log_a x$ are equivalent
3.) the two properties below

This comes from the fact that logs and exponentials are inverses.

Base-Exponent Property: For any $a > 0$ and $a \neq 1$, we know that $a^x = a^y$ if and only if $x = y$.

So, if $5^x = 5^4$, then what must be true?

$x = 4$

Recall "if and only if" means the logic flows both ways.

Logarithmic Equality Property: For any $M > 0$, $N > 0$, $a > 0$, and $a \neq 1$, we know that $M = N$ if and only if $\log_a M = \log_a N$.

So, if $\log_2 x = \log_2 10$, then what must be true?

These two properties are given for a general base a but we will see them primarily with that value being e.

expl 6: Algebraically solve. Round to three decimal places.

a.) $e^{-5x} = 250$

Log Equality Prop:
(take ln of both sides)

$\ln e^{-5x} = \ln 250$

(prop 6, page 4)

$-5x = \ln 250$

$x = \frac{\ln 250}{-5}$

$x \approx -1.104$

b.) $6\ln(2x) = 18$

$\frac{6}{} \quad \frac{6}{}$

$\ln(2x) = 3$

$\log_e(2x) = 3$

$e^3 = 2x$

$x = e^3/2$

$x \approx 10.043$

$y = \log_a x \Leftrightarrow a^y = x$ pg 5

Application: Compound Interest:

We are going to invest an amount of money (**principal, present value**) into a bank account that pays **compound interest**. We will find the amount of money at the end of some time period (**accumulated** or **future value**). Sometimes, we frame this as someone **borrows** P dollars and is charged compound interest.

You may have studied **simple interest** in the past.

With simple interest, interest is based on how much was initially deposited. But after one year, we have an extra \$50 (the interest earned or 5% of \$1,000) that we could invest. *If* we put that money back into the account and let it earn interest, we call that **compound interest**. All interest that is earned is subsequently reinvested to start earning the same rate as the initial deposit. The formula for that is here.

Compound Interest Formula:

$$A = P\left(1+\frac{r}{n}\right)^{n\cdot t}$$

P = initial deposit (principal)
A = amount in account after t years
r = annual interest rate (decimal form)
n = number of compoundings per year
t = number of years

This assumes the account is compounded in one of several common ways. We could compound an account yearly (n = 1 time a year), semiannually (n = 2 times a year), quarterly (n = 4), monthly (n =12), weekly (n = 52), or daily (n = 365).

And then, there's compounding continuously. Here, we imagine that n gets larger and larger; that is, $n \to \infty$. Limits help us derive the formula, which actually becomes simpler.

Continuously Compounded Interest Formula:

$$A = Pe^{r \cdot t}$$

The variables mean the same but notice we are using the irrational number e.

expl 7: Marcos wants \$2500 to go on vacation. He has \$1500 he will invest in an account that will compound *continuously* at 5.26% to earn his vacation fund. How many years (to the nearest tenth) will it take?

? $t = ?$

$2500 = 1500e^{0.0526t}$

$\frac{5}{3} = e^{0.0526t}$

$\ln(5/3) = \ln e^{0.0526t}$ ↓(prop 6, pg 4)

$\ln 5/3 = 0.0526t$

$t = \frac{\ln(5/3)}{0.0526}$

$t \approx 9.7$ years

expl 8: Consider a compound interest account that compounds its interest *monthly*. Find the future value if \$5,000 is invested at 3.15% interest for 12 years.

$A = ?$

$A = P(1 + r/n)^{nt}$

$n = 12$

$= 5000(1 + 0.0315/12)^{12(12)}$

$A \approx \$7293.20$

How does the formula change if it compounds daily or weekly?

expl 9: Consider a compound interest account that compounds *continuously*. Find the doubling time when \$5,000 is invested at 3.15% interest.

$t = ?$

need A to be double the principal

$A = Pe^{rt}$

$10{,}000 = 5000e^{0.0315t}$

$2 = e^{0.0315t}$

$\ln 2 = \ln e^{0.0315t}$

$\ln 2 = 0.0315t$

$t = \frac{\ln 2}{0.0315}$

$t \approx 22$ years