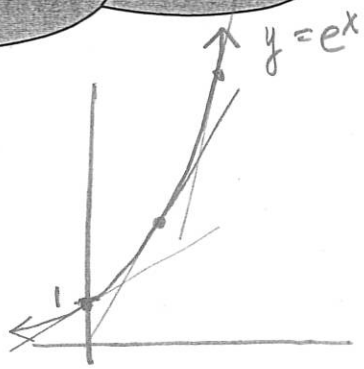
Business Calculus
Class Notes

The natural exponential function was jogging alone down a dark alley. Suddenly, a differential operator jumps from the shadows. Was our function worried? She was *not* as she knew he could do *nothing* to her!

Section 2.2: Derivatives of Natural Exponential Functions

Recall that the **natural exponential function** is $f(x) = e^x$.
Draw a quick graph and draw in a few tangent lines.
We are interested in the **rate of change** for this natural exponential function. Derivatives, ho!



The book does a good job of explaining this next gem.

Derivatives of Natural Exponential Functions:
A fun little tidbit about this **natural exponential function** is that it is its own derivative. Rather,

$$\frac{d}{dx}\left(e^x\right) = e^x$$

Easiest rule you'll ever get.

But what if this exponent is *not* just x, but rather a function in x? Then, we'd need the Chain Rule. So, we have this lovely nugget.

$\frac{d}{dx}(g(f(x))) = g'(f(x)) \cdot f'(x)$

$$\frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} \cdot f'(x)$$

uses Chain Rule where $g(x) = e^x$

Equivalently, if we think of this exponent as u which is a function in x, we would write this.

$$\frac{d}{dx}\left(e^u\right) = e^u \cdot \frac{du}{dx}$$

like $u(x) = 3x^2 + 4x$

Let's do some examples.

expl 1: Find the derivative of $y = 4e^x + 3x^2$.

constant multiple rule (1.5)

$\frac{d}{dx}(e^x) = e^x$

Go term by term.

$\frac{dy}{dx}$ or $\frac{d}{dx}(y)$ or y'

$y' = 4e^x + 6x$ ← Constant multiple rule + Power Rule

$y' = 4e^x + 6x$

Imagine $y'' = 4e^x + 6$

$y''' = 4e^x + 0$

$g(x) = e^x$

$f(x) = 3x^2+4x$ $\Rightarrow g(f(x)) = e^{3x^2+4x}$

expl 2: Find the following.

a.) $\frac{d}{dx}\left(e^{3x^2+4x}\right) = e^{3x^2+4x}(6x+4)$

OR $= (6x+4)e^{3x^2+4x}$

b.) $\frac{dy}{dt}$ for the function $f(t) = e^t + \sqrt{5e^{3t}}$ $(5e^{3t})^{1/2}$

OR $f'(t)$

Two instances of the Chain Rule!

$\frac{dy}{dt} = e^t + \frac{1}{2}(5e^{3t})^{-1/2}\left(\frac{d}{dt}(5e^{3t})\right)$ Chain Rule

Where do you stop?

$= e^t + \frac{1}{2}(5e^{3t})^{-1/2} \cdot 5e^{3t}\cdot\left(\frac{d}{dt}(3t)\right)$ Chain Rule

$= e^t + \frac{1}{2}(5e^{3t})^{-1/2}\cdot 5e^{3t}\cdot 3$

$= e^t + \frac{15}{2}e^{3t}(5e^{3t})^{-1/2}$

$\frac{d}{dx}(fg) = f'g + fg'$

c.) $\frac{d}{dx}\left(5e^{3x}\cdot x^6\right)$ $f(x) = 5e^{3x}$ $g(x) = x^6$

Do you see a product?

$= (5e^{3x}\cdot 3)x^6 + 5e^{3x}\cdot 6\cdot x^5$ ← Product Rule

$f' \cdot g + f \cdot g'$

Combine like terms. Simplify as you go.

$= 15x^6e^{3x} + 30x^5e^{3x}$

d.) $f''(x)$ for the function $f(x) = (x^4+8x^3+7)e^x$

Product Rule

$f'(x) = (4x^3+24x^2)e^x + (x^4+8x^3+7)\cdot e^x$

$f'(x) = (x^4+12x^3+24x^2+7)\cdot e^x$

$f''(x) = (4x^3+36x^2+48x)e^x + (x^4+12x^3+24x^2+7)e^x$

$f''(x) = e^x(x^4+16x^3+60x^2+48x+7)$

$\frac{d^2y}{dx^2}$

Application: Compound Interest: 0.0315 (decimal form)

expl 3: Consider a compound interest account that compounds *continuously* with $5,000 invested at 3.15% interest. Answer the following questions. Include units.

a.) What is the formula for the amount in the account after t years?

$A = Pe^{rt} \rightarrow A = 5000\,e^{0.0315t}$ (dollars)

b.) What is the rate of change for the amount in the account after t years?

$A' = 5000\,e^{0.0315t} \cdot 0.0315$ ← der of exponent

$A' = 157.5\,e^{0.0315t}$

c.) What is the rate of change for this rate of change for the amount in the account after t years? Don't round.

$A'' = 157.5\,e^{0.0315t} \cdot 0.0315$ ← der of exponent

$A'' = 4.96125\,e^{0.0315t}$

d.) Evaluate both rates at 10 years. Round to two decimal places. (Do you see why?) Include units. Interpret.

$A'\big|_{t=10} = 157.5\,e^{0.0315(10)} \approx \$215.82/\text{year}$

At the 10 year mark, the amt. of money is increasing at $215.82/year.

$A''\big|_{t=10} = 4.96125\,e^{0.0315(10)} \approx \$6.80/\text{year/year}$ or $\$/\text{year}^2$

At the 10-year mark, the rate of change for the amt of money in account is increasing at the rate of $6.80/year every year.

e.) The original function has a point at (10, 6851.30). What does this mean? Find the equation of the tangent line to the curve at this point. Graph both on your calculator to check.

$A = 5000\,e^{0.0315(10)}$

$A \approx 6851.30$

After 10 years, we have $6851.30 in account.

$[0, 25] \times [0, 10{,}000]$

(1.4) Tangent line eqn:

$y - f(a) = f'(a)(x-a)$

$y - 6851.30 = 215.82(x-10)$

$y = 215.82x + 4693.10$

Eqn of tangent line

