Can you picture the graph of $y=\ln(x)$?
What is its rate of change?

Business Calculus
Class Notes
Section 2.3: Derivatives of Natural Logarithmic Functions

The natural logarithmic function (usually written as $y=\ln(x)$ which means $y=\log_e(x)$) may be involved in problems where we want to know the function's rate of change. So, what is the derivative of $y=\ln(x)$? The book has a quick proof of this but we just give the rule here.

Derivatives of Natural Logarithmic Functions:
For any positive number x (because the function's *not* defined otherwise), we have

$$\frac{d}{dx}(\ln x)=\frac{1}{x}$$

But what if the thing we're taking the natural log of is *not* just x, but rather a function in x? Then, we'd need the Chain Rule. So, we have this lovely nugget.

$$\frac{d}{dx}(\ln f(x))=\frac{1}{f(x)}\cdot f'(x)=\frac{f'(x)}{f(x)}$$

Equivalently, if we think of the argument of the log function as u which is a function in x, we would write this.

$$\frac{d}{dx}\ln u=\frac{1}{u}\cdot\frac{du}{dx}$$

The **argument** of a function is its input.

Let's do some examples.

Ooh, product rule! $\frac{d}{dx}(fg)=f'g+fg'$

What's $\frac{d}{dx}x^2$? What's $\frac{d}{dx}\ln x$?

expl 1: Find the derivative of $y=x^2\ln x+4$.

$\frac{dy}{dx}$ or $\frac{d}{dx}(y)$?

product rule

$$y'=2x\cdot\ln x+\frac{x^2}{1}\cdot\frac{1}{x}+\frac{d}{dx}(4)$$

$$y'=2x\ln x+x+0$$

$$y'=2x\ln x+x$$

$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$ $\quad\|\quad \frac{d}{dx}(f/g) = \frac{f'g - fg'}{g^2}$ (1.6)

2. Find the derivative of $y = \ln\left(\frac{x+4}{x^2-8}\right)$. We will do this using two different methods to explore our options.

This is $f(x) = \frac{x+4}{x^2-8}$

a.) Method 1: Quotient rule to find $\frac{d}{dx}\left(\frac{x+4}{x^2-8}\right)$ because $\frac{d}{dx}\ln u = \frac{1}{u}\cdot\frac{du}{dx}$

$$\frac{d}{dx}\left(\ln\left(\frac{x+4}{x^2-8}\right)\right) = \frac{1}{\frac{x+4}{x^2-8}} \cdot \frac{d}{dx}\left(\frac{x+4}{x^2-8}\right)$$

Quotient Rule

flip and multiply

$$= \frac{x^2-8}{x+4} \cdot \frac{1(x^2-8) - (x+4)(2x)}{(x^2-8)^2}$$

$$= \frac{(x^2-8)}{x+4} \cdot \frac{x^2-8-2x^2-8x}{(x^2-8)^2}$$

$$= \frac{-x^2-8x-8}{(x+4)(x^2-8)}$$

$$= \frac{-1(x^2+8x+8)}{(x+4)(x^2-8)}$$

b.) Method 2: Logarithm Properties (specifically, $\ln\left(M/N\right) = \ln M - \ln N$) to avoid Quotient Rule

$$\ln\left(\frac{x+4}{x^2-8}\right) = \ln(x+4) - \ln(x^2-8)$$

$$\frac{d}{dx}\left(\ln\left(\frac{x+4}{x^2-8}\right)\right) = \frac{d}{dx}(\ln(x+4)) - \frac{d}{dx}(\ln(x^2-8))$$

$f(x) = x+4$, der $= \frac{f'(x)}{f(x)}$

$f(x) = x^2-8$, der $= f'(x)/f(x)$

$$= \frac{1}{x+4} - \frac{2x}{x^2-8}$$

To verify it's the same:

You should be able to show your answers are equivalent.

$$\frac{1}{x+4} - \frac{2x}{x^2-8}$$

$$= \frac{1(x^2-8)}{(x+4)(x^2-8)} - \frac{2x(x+4)}{(x^2-8)(x+4)} = \frac{x^2-8-2x^2-8x}{(x+4)(x^2-8)} = \frac{-x^2-8x-8}{(x+4)(x^2-8)}$$ (which is what we got above) ✓

The Chain Rule (or Extended Power Rule) says

$$\frac{d}{dx}\left(g(x)\right)^k = k\cdot\left(g(x)\right)^{k-1}\cdot\frac{d}{dx}g(x)$$

3. Find the derivative of $f(x)=(\ln x)^5$.

$g(x)=\ln x$

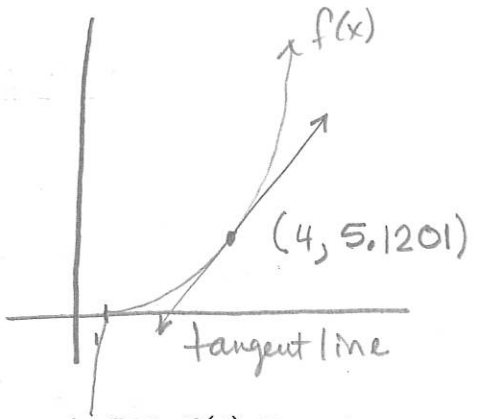
$f'(x)$ or $\frac{dy}{dx} = 5\cdot(\ln x)^4\cdot\frac{1}{x}$ ← $\frac{d}{dx}(\ln x)=\frac{1}{x}$

$f'(x) = \frac{5}{x}(\ln x)^4$

Not $\ln(x^4)$

4. Find the equation of the tangent line to the graph of $f(x)=(\ln x)^5$ at $x=4$. Follow these steps.

a.) Graph the function and copy it here. The Standard window will be fine. Use the Value function on your calculator to find $f(4)$ to four decimal places. On your paper, plot and label the point at $x=4$ and sketch the tangent line.



b.) Find the slope of this tangent line at this point. In other words, find $f'(4)$. Round to four decimal places.

$f'(x) = \frac{5}{x}(\ln x)^4$ $\quad f'(4)\approx 4.6167$

$f'(4) = \frac{5}{4}(\ln 4)^4$

c.) Recall that the equation of this tangent line (at $x=a$) is always $y-f(a)=f'(a)(x-a)$. Put in what you know to find the equation of this line. Write your final answer in slope-intercept form.

$y-f(4)=f'(4)(x-4)$

$y-5.1201 = 4.6167(x-4)$

$y-5.1201 = 4.6167x-18.4668$

$y = 4.6167x-13.3467$

Graph this line with the original function to check.

Recall: Rates of Change, in General:

We are working with specific examples here but understand that these ideas will work for any function. Let's say you know a function $p(t)$ measures something like a population *or* price per item *or* number of internet users, etc. Here, t is taken to be time where $t \geq 0$. This function's **instantaneous rate of change** is its derivative $p'(t)$ or $\frac{dp}{dt}$. This measures how fast $p(t)$ is changing with respect to time. The units of this rate of change will always be "units of $p(t)$ divided by units of t".

Application: Marginal Cost, Profit, and Revenue:

Recall that profit equals revenue minus cost.

Let $C(x)$ be the cost of producing x items. Perhaps we will be given the function $C(x) = -0.02x^2 + 50x + 100$ dollars.

The **Marginal Cost** is the approximate cost to produce *one additional item* after producing x items. And that sounds like rate of change. Hence, marginal cost is $C'(x)$.

Further, let $p(x)$ denote the sale price per item if x items are sold.

Notice this is *lowercase p*.

We would find **Revenue** (which is total money brought in) as the product $R(x) = x \cdot p(x)$.

Which leaves us **Profit** (the money we really made) as

$$P(x) = R(x) - C(x)$$
$$= x \cdot p(x) - C(x)$$

Here's the *uppercase P*.

We are in the US so our units for cost, revenue, and profit will usually be in dollars with the units of their derivatives derived accordingly.

It is *not* a long stretch to define marginal revenue and profit as you see here.

The **Marginal Revenue,** $R'(x)$, is the approximate revenue gained or lost by selling *one additional item* after selling x items.

The **Marginal Profit,** $P'(x)$, is the approximate profit gained or lost by selling *one additional item* after selling x items.

pg 5 (2 notes)

$x = e^y \Leftrightarrow y = \log_e x$ (or $y = \ln x$)

$p(x)$ = price per item ($)

x = number of shirts (in 1000's)

expl 5: The demand for a shirt can be modeled by $p(x) = 65.9 - 10.1 \ln x, \quad 0 \le x \le 680$. Here, $p(x)$ is the price wholesalers will pay, in dollars, and x is the number of shirts that will be sold, in thousands. Answer the following questions.

a.) If the price wholesalers are willing to pay is \$20, how many shirts would be sold? Play with solving it algebraically and graphically.

$P(x) = 65.9 - 10.1 \ln x$

$20 = 65.9 - 10.1 \ln x$

$-45.9 = -10.1 \ln x$

$4.54 \approx \ln(x)$

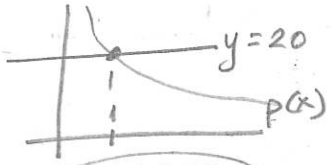
$x = e^{4.54} \rightarrow x \approx 94.118$ thousands of shirts

OR

$20 = 65.9 - 10.1 \ln x$

$y_1 = 65.9 - 10.1 \ln x$

$y_2 = 20$



$x \approx 94.118$

OR

94,118 shirts

b.) Find a function for the revenue $R(x)$ of this shirt.

Keep in mind that $R(x) = x \cdot p(x)$.

$R(x) = x \cdot p(x)$

$= x(65.9 - 10.1 \ln x)$

$R(x) = 65.9x - 10.1x \ln(x)$

c.) Find a function for the marginal revenue which is $R'(x)$.

$R(x) = 65.9x - 10.1x \ln x$ $\quad \frac{d}{dx}(fg) = f'g + fg'$

$R'(x) = 65.9 - (10.1 \ln x + 10.1x(1/x))$

$= 65.9 - 10.1 \ln x - 10.1$

$R'(x) = 55.8 - 10.1 \ln x$

Keep in mind that x is in thousands.

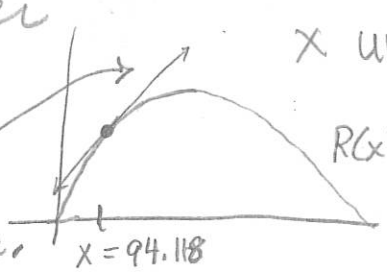
d.) Find and interpret $R'(94.118)$. Include a graph of $R(x)$ and mark the point where $x = 94.118$.

$R'(94.118) = 55.8 - 10.1 \ln(94.118)$

$\approx \$9.90$ / thousand shirts

Meaning, when $x = 94.118$ thousands of shirts, $R(x)$ is changing (or increasing) at the rate of 9.90 dollars per thousand shirts.

$R'(94.118)$ is the slope of this tangent line.



Start off with xmin = 0 and xmax = 680. Use the Value function to find some value of revenue that will help you set ymax.

R(x) units: dollars

x units: thousands of shirts

2nd derivative

expl 6: Find $\frac{d^2y}{dx^2}$ for the function $y = x^2 \ln x + 4$.

We found the first derivative for example 1.

y' or $\frac{dy}{dx} = 2x \ln x + \frac{x^2}{1} \cdot \frac{1}{x} + 0$ $\quad \frac{d}{dx}(fg) = f'g + fg'$

$\frac{dy}{dx} = 2x \ln x + x$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2 \cdot \ln x + \frac{2x}{1} \cdot \frac{1}{x} + 1$

Product Rule Again!

$= 2 \ln x + 2 + 1$

$\frac{d^2y}{dx^2} = 2 \ln x + 3$