The first derivative of a function can tell you a lot about its graph.

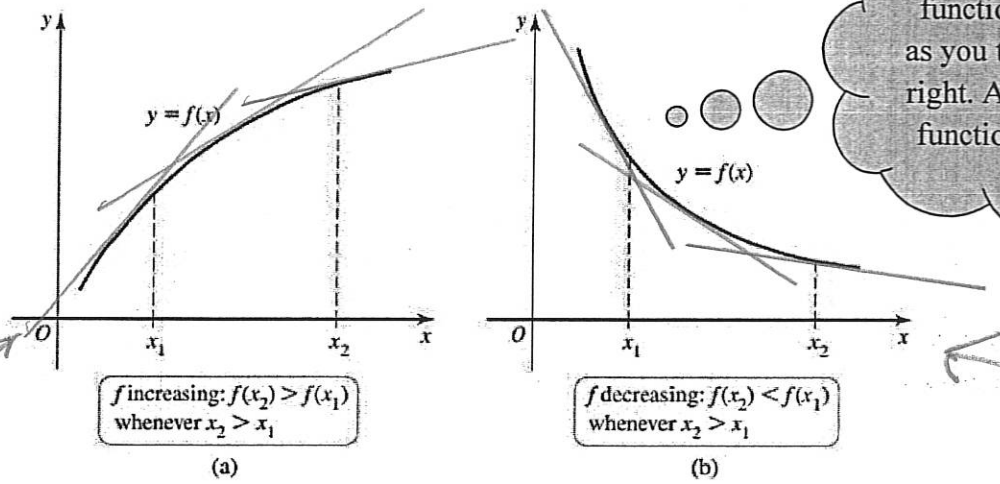
Business Calculus
Class notes
Section 3.1: Applications of Derivatives: The First Derivative Test

The concepts of increasing and decreasing graphs come back from algebra here. We will understand how the first derivative helps us determine these important characteristics of a function's graph. First, let's review.

DEFINITION Increasing and Decreasing Functions

Suppose a function f is defined on an interval I. We say that f is **increasing** on I if $f(x_2) > f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$. We say that f is **decreasing** on I if $f(x_2) < f(x_1)$ whenever x_1 and x_2 are in I and $x_2 > x_1$.

An increasing function will rise as you travel left to right. A decreasing function will fall.



As you look at these graphs, imagine some tangent lines. Go crazy and draw a few in. What do you notice about those of an increasing graph versus those tangent lines on the decreasing graph?

Increasing graph has tangent lines with positive slopes.

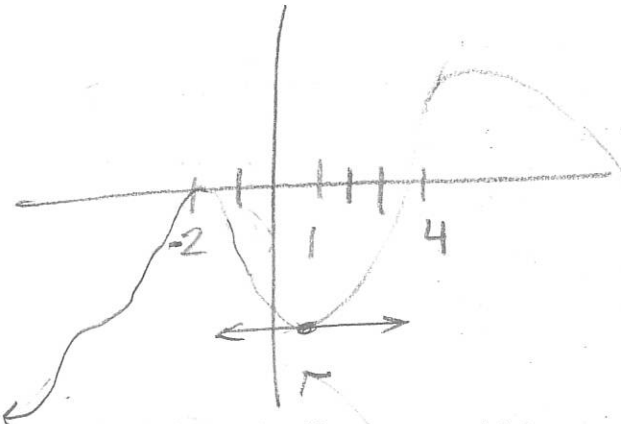
Decreasing graph has tangent lines with negative slopes.

This realization brings us to our first theorem of this section. Your examples should be pretty good evidence of its truth. (I liked how another book stated this stuff so I am using those.)

THEOREM 4.7 Test for Intervals of Increase and Decrease
Suppose f is continuous on an interval I and differentiable at all interior points of I. If $f'(x) > 0$ at all interior points of I, then f is increasing on I. If $f'(x) < 0$ at all interior points of I, then f is decreasing on I.

expl 1: Sketch a graph of a function $f(x)$ such that $f'(x) < 0$ on $(-2, 1)$ and $(4, \infty)$ but $f'(x) > 0$ on $(-\infty, -2)$ and $(1, 4)$.

Start with an xy-plane and mark off x-values.



On homework, you will see these questions as multiple-choice. What if you were told that $f'(-2) = 0$? What would that tell you about the function's graph?

↳ The tangent line would have a slope of 0 which means it's horizontal. And we may have a min or max there.

expl 2: Find intervals on which f is increasing or decreasing.
$f(x) = x^4 - 4x^3 + 4x^2$

Dust off your factoring skills.

$f'(x) = 4x^3 - 12x^2 + 8x$

$0 = 4x^3 - 12x^2 + 8x$

$0 = 4x(x^2 - 3x + 2)$

$0 = 4x(x-1)(x-2)$

$x = 0$ or $x - 1 = 0$ or $x - 2 = 0$

$x = 1$ $x = 2$

test $x = -2$ $x = 1/2$ $x = 3/2$ $x = 5$

0 1 2

$f' < 0$ $f' > 0$ $f' < 0$ $f' > 0$

See if $f'(x) < 0$ or $f'(x) > 0$ for these intervals $f'(x) = 4x(x-1)(x-2)$

$f'(-2) = \text{neg} \cdot \text{neg} \cdot \text{neg} = \text{neg}$

We'll make a "sign graph".

$f'(\frac{1}{2}) = \text{pos} \cdot \text{neg} \cdot \text{neg} = \text{pos}$

$f'(3/2) = \text{pos} \cdot \text{pos} \cdot \text{neg} = \text{neg}$

$f'(5) = \text{pos} \cdot \text{pos} \cdot \text{pos} = \text{pos}$

So, $f(x)$ decreases on the intervals $(-\infty, 0)$ and $(1, 2)$ and $f(x)$ increases on the intervals $(0, 1)$ and $(2, \infty)$.

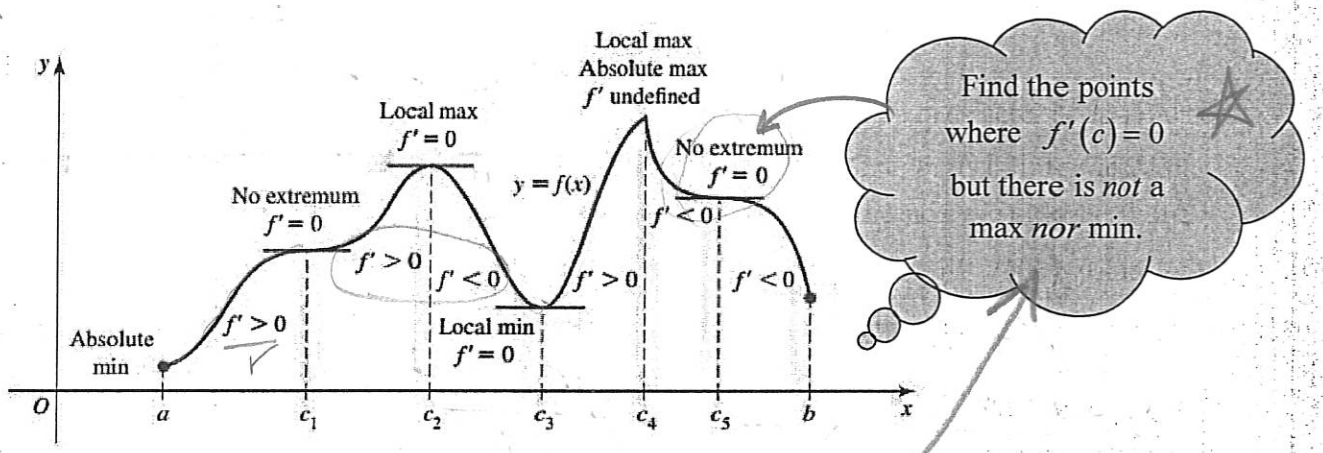
Sign Graphs:

The process, as seen in the previous example, will be to find critical points (where $f'(x)=0$ or is undefined) and then to fill out a **sign graph** using these values as endpoints of intervals. For each interval, we need only find out if $f'(x)$ is positive or negative. Hence, we only need one value in each interval to test. If one $f'(x)$ is positive in the interval, then *all* $f'(x)$ values will be. You can always graph $f(x)$ on the calculator to check your conclusions.

Definition: Critical value of a function: Any x-value c in the domain of f such that $f'(c)=0$ or $f'(c)$ is undefined. The book further defines a **critical point** to be, in ordered pair notation, $(c, f(c))$.

Local Minimums and Maximums:

A critical point *may* be where a local min or max occurs but it does *not* have to be. Check out this graph where we have both $f'(c)=0$ and $f'(c)$ is undefined.



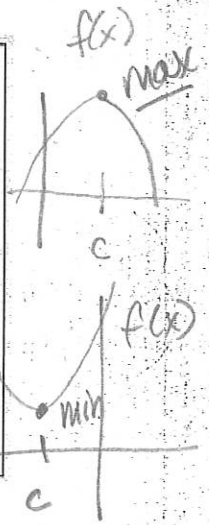
Find the local maxes and mins above. Do you notice something interesting happening on either side of those extrema? We have the **First Derivative Test** to put it in words.

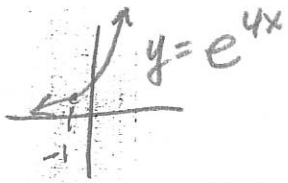
THEOREM 4.8 First Derivative Test

Assume f is continuous on an interval that contains a critical point c, and assume f is differentiable on an interval containing c, except perhaps at c itself.

- If f' changes sign from positive to negative as x increases through c, then f has a **local maximum** at c.
- If f' changes sign from negative to positive as x increases through c, then f has a **local minimum** at c.
- If f' is positive on both sides near c or negative on both sides near c, then f has no local extreme value at c.

as you go left to right





$$\frac{d}{dx}(fg) = f'g + fg'$$

expl 3: Find the critical values for the function below. Then use a sign graph to find relative extrema.

$h(x) = x \cdot e^{4x}$

First step: find the derivative!

Give relative extrema in ordered pair notation.

$h'(x) = 1 \cdot e^{4x} + x \cdot 4 \cdot e^{4x}$

$h'(x) = e^{4x}(1 + 4x)$

where is $h'(x)$ undefined? (nowhere)

where is $h'(x) = 0$?

$0 = e^{4x}(1+4x)$

$e^{4x} = 0$ or $1 + 4x = 0$

(no soln)

$4x = -1$

$x = -1/4$

Test $x = -1$ | $x = -1/4$ | $x = 0$

$h' < 0$ | $h' > 0$

$h'(x) = e^{4x}(1+4x)$

$h'(-1) = e^{-4}(1+4(-1)) = \text{neg}$

$h'(0) = e^{0}(1+0) = \text{pos}.$

The 1st Der. Test tells us $h(x)$ has a local minimum at $(-0.25, -0.09)$.

$h(-0.25) \approx -0.09$

expl 4: Find the critical values for the function below. Then use a sign graph to find relative extrema.

$g(x) = \ln(x^2 + 4x + 10)$

from 2.3 $\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$

$g'(x) = \frac{2x+4}{x^2+4x+10}$

where's $g'(x)$ undefined? (find where bottom $= 0$)

$0 = x^2 + 4x + 10$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(10)}}{2 \cdot 1}$

$x = \frac{-4 \pm \sqrt{-24}}{2}$

negative under radical!

So, $0 = x^2 + 4x + 10$ has no real solns.

→ No critical values here.

where's $g'(x) = 0$? (find where top $= 0$)

$0 = 2x + 4$

$-4 = 2x$

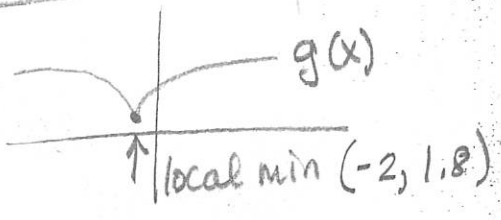
$-2 = x$ Critical value

Test $x = -4$ | $x = -2$ | $x = 0$

$g' < 0$ | $g' > 0$

$g'(-4) = \frac{\text{neg}}{\text{pos}} = \text{neg}$ $g'(0) = \frac{\text{pos}}{\text{pos}} = \text{pos}$

There's a local minimum at $(-2, \ln 6)$.



find where $g'(x) = 0$ or undefined

expl 5a: Find the critical values for the function below. Then use a sign graph to find relative extrema.

$g(x) = x^3 - 3x - 4$

$g'(x) = 3x^2 - 3$

(undefined nowhere)

$0 = 3x^2 - 3$

$3 = 3x^2$

$1 = x^2$

$x = \pm 1$ critical values

Test $x=-2$, $x=0$, $x=2$

$g'>0$ | -1 | $g'<0$ | 1 | $g'>0$

$g'(-2) = $ pos ↑ $g'(0) = -3$ neg ↑ $g'(2) = $ pos

★ ★

$g(-1) = -2$

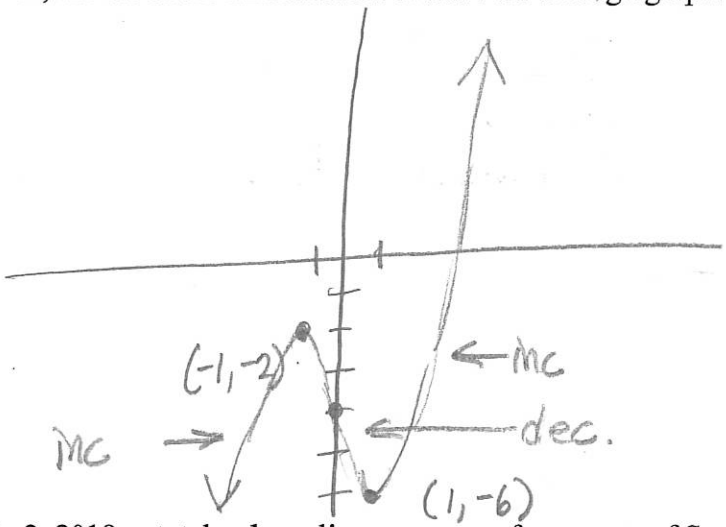
Local max at $(-1, -2)$.

Local min at $(1, -6)$.

expl 5b: One big reason we find relative extrema is to help graph the function by hand. Use the above results to write the intervals where the function $g(x) = x^3 - 3x - 4$ is increasing or decreasing. Then, use all of the information to draw out a rough graph.

$g(x) = x^3 - 3x - 4$

$g(0) = -4$



expl 6: On July 2, 2019, a total solar eclipse was seen from parts of South America. The path of the eclipse was modeled by $f(t) = 0.00259t^2 - 0.457t + 36.237$ where $f(t)$ is the latitude in degrees south of the equator at t minutes after the start of the total eclipse. What is the latitude closest to the equator, in degrees, in which the eclipse was visible?

Minimum = ?

Find $f'(t) = 0.00518t - 0.457$ and solve $f'(t) = 0$

$0 = 0.00518t - 0.457$

$0.457 = 0.00518t$

$t \approx 88$ minutes

$f(88)$

The minimum occurs at $(88, 16)$.

The latitude closest to the equator where the eclipse is visible is $16°$ South.

Do you remember what quadratic functions look like?

This opens up because $0.00259 > 0$

(or $a > 0$ for $y = ax^2 + bx + c$)

There's a minimum.

Calculator Usage and Algebra:

In algebra, we might use the **Maximum** and **Minimum** Functions under the **Calculate** menu (second function of the TRACE button) to find these relative extrema. Here, we will rely on algebraic means. However, feel free to check on the calculator.

★ expl 7: From algebra class, do you remember the formula for the x-value of the vertex of a quadratic function in the form $y = ax^2 + bx + c$? Can you verify it using calculus?

vertex x value: $x = -b/2a$

Verify: $y = ax^2 + bx + c$

$y' = 2ax + b + 0$

$y' = 2ax + b$

(This is undefined nowhere. But $y' = 0$...)

$0 = 2ax + b$

$-b = 2ax$

$\frac{-b}{2a} = x$ ✓