The second derivative tells us when the first derivative is increasing or decreasing. What would that say about the function itself?

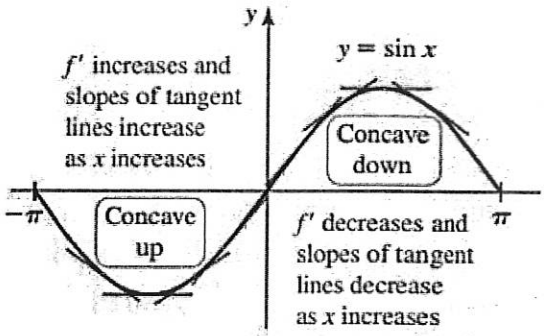
Business Calculus
Class notes
Section 3.2: Applications of Derivatives: Concavity and the Second Derivative Test

The first derivative tells us where a function is increasing or decreasing. Hence, the second derivative would tell us where the first derivative in increasing or decreasing. What does that say about the original function?

Concavity and Inflection Points:

Take a look at this graph. Algebra taught us the concepts of concave up and down. Now, we apply calculus to this graph.

There is a point in the middle where the graph changes from concave up to concave down. This point is an **inflection point**. We will also see that the slopes of these tangent lines stop increasing and start decreasing.



Below, we see how the derivative plays a role.

DEFINITION Concavity and Inflection Point

Let f be differentiable on an open interval I. If f' is increasing on I, then f is **concave up** on I. If f' is decreasing on I, then f is **concave down** on I.

If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

Now, this mentions that "If f' is increasing [or decreasing] on I …"

How does calculus tell us when a function is increasing or decreasing? That, you say, involves finding the function's derivative. What is the derivative of f'? And so, we have this test for concavity.

THEOREM 4.10 Test for Concavity

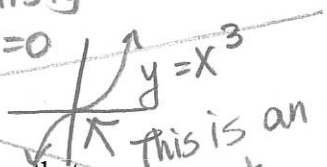
Suppose f'' exists on an open interval I.

- If $f'' > 0$ on I, then f is concave up on I.
- If $f'' < 0$ on I, then f is concave down on I.
- If c is a point of I at which f'' changes sign at c (from positive to negative, or vice versa), then f has an inflection point at c.

$y = x^2$, $y' = 2x$, $y'' = 2$

$y = x^3$, $y' = 3x^2$, $y'' = 6x$

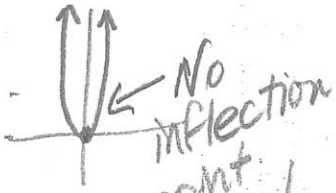
This would satisfy $y'' = 0$ for $x = 0$



Do *not* read too much into this. It is *not necessarily true* that if $f''(c) = 0$, then f'' *must* have changed sign and you would find an inflection point. Can you think of an example where $f''(c) = 0$ for some c but f has *no* inflection point there? (Try out $y = x^n$ for various values of n.)

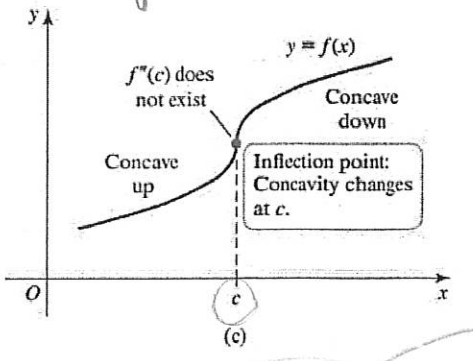
$y = x^4$, $y' = 4x^3$, $y'' = 12x^2$

This $y'' = 0$ for $x = 0$



When the Second Derivative Does *Not* Exist:

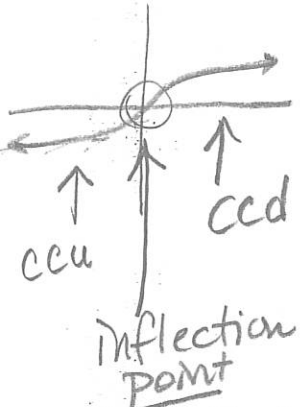
★ An inflection point may also occur when the second derivative does *not* exist. Here is the graph of such a function.



(c)

Another example is the function $f(x) = \sqrt[3]{x} = x^{1/3}$. Do you have its graph in your head?

Find the second derivative of $f(x) = \sqrt[3]{x} = x^{1/3}$ to show it does *not* exist at $x = 0$ (its inflection point).



$f(x) = x^{1/3}$

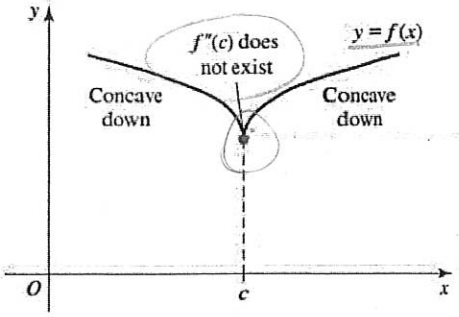
$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)x^{-5/3}$

$f''(x) = \frac{-2}{9x^{5/3}}$ Notice $f''(0)$ dne.

$-\frac{2}{3} - 1 = -\frac{2}{3} - \frac{3}{3} = -\frac{5}{3}$

Here's another example of a function whose second derivative does *not* exist.



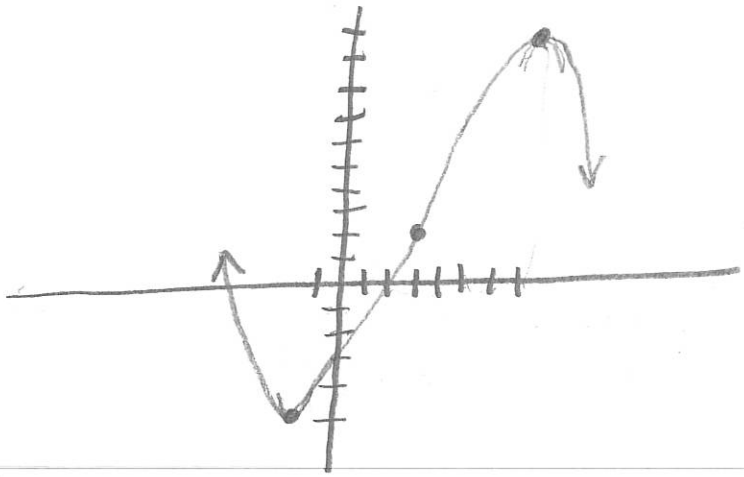
(d)

Here, we do *not* call the point in the middle an inflection point. Do you know why?

Both sides of $x = c$ are concave down.

expl 1: Sketch a possible graph that possesses these characteristics. Use the table below to organize your thoughts. Answers may vary.

$f'(-1) = 0, f''(-1) > 0, f(-1) = -5,$
$f''(3) = 0, f(3) = 2,$
$f'(7) = 0, f''(7) < 0, f(7) = 10$



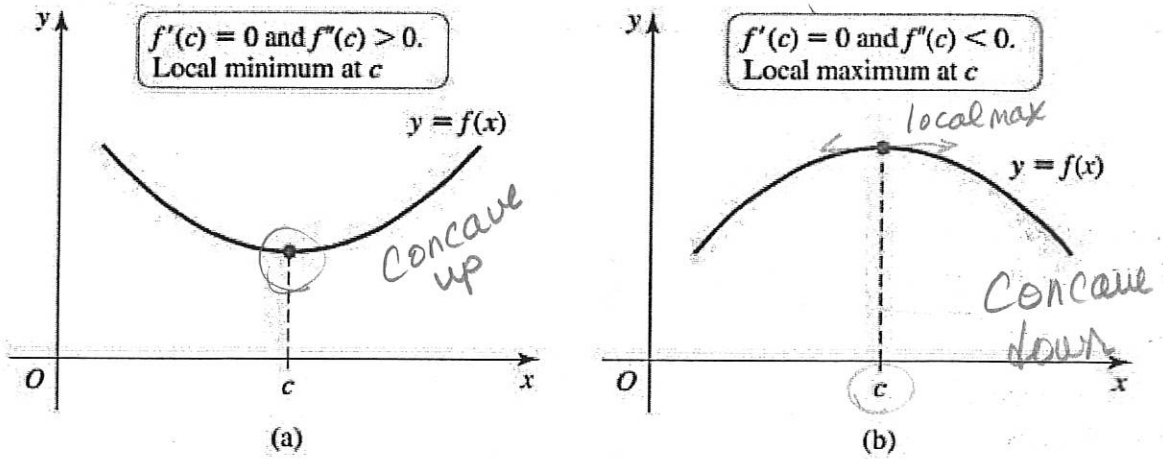
x-value	function value $f(x)$	horizontal tangent line?	increasing /decreasing?	concave up/down?
-1	-5	yes	?	ccu
3	2	?	?	inflection pt?
7	10	yes	?	ccd

Some tests will be inconclusive.

When we drew the part of the graph at $x = -1$ and $x = 7$, we wanted to show a horizontal tangent line and concavity. That resulted in a relative minimum and maximum, respectively. Was that inevitable?

Second Derivative Test:

You may remember from algebra the connection between concavity and local maxes and mins. Take a look at these pictures, noticing the left graph is concave up and the right is concave down.



The Second Derivative Test (proven, in part, with the First Derivative Test) is our last piece of this puzzle.

THEOREM 4.11 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval containing c with $f'(c) = 0$.

- If $f''(c) > 0$, then f has a local minimum at c (Figure 4.38a).
- If $f''(c) < 0$, then f has a local maximum at c (Figure 4.38b).
- If $f''(c) = 0$, then the test is inconclusive; f may have a local maximum, a local minimum, or neither at c.

If case 3 happens, you may be able to use the First Derivative Test.

Recall: Definition: Critical value of a function: Any x-value c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined.

The **critical points** are very important to us.

The **Second Derivative Test** tells us to find $f''(x)$ for those values of x that make the first derivative zero. If that second derivative is *not* zero, it gives us good information.

But what if $f''(x) = 0$ or does *not* exist for some x-value? It gives us a point of interest that we need to investigate more. Consider this theorem.

Theorem 6

Finding Points of Inflection

If f has a point of inflection, then it must occur at a value x_0, in the domain of f, where

$f''(x_0) = 0$ or $f''(x_0)$ does not exist.

Again, as mentioned before, do *not* read too much into this. It is *not necessarily true* that if $f''(c) = 0$, then f'' *must* have changed sign and you would find an inflection point.

Logic Sidebar:

A **conditional** statement is generalized as "If p, then q.". Its **converse** is "If q, then p." The converse of a statement is *not* necessarily true when a statement is true.

We take Theorem 6 above as the original statement. To assume the logic works backwards and "If $f''(c) = 0$ or does *not* exist, then there would be an inflection point at c." is *incorrect*.

Homework problems will require that you use the First Derivative Test and Tests for Intervals of Increase or Decrease from the previous section in addition to what we learn in this section. We'll focus here on what is new.

expl 2: Determine intervals on which the following function is concave up or concave down. Identify inflection point(s). Give points in ordered pair notation.

$g(x) = x^3 - 6x^2 + 9x + 1$

Find g'' and look back to page 1. Use a sign graph.

$g' = 3x^2 - 12x + 9$

$g'' = 6x - 12$

Set $g'' = 0 = 6x - 12$

$12 = 6x$

$2 = x$

Test $x=0$ | $x=3$ | 2

$g''<0$ ccd | $g''>0$ ccu

(2,3)

The fnc $g(x) = x^3 - 6x^2 + 9x + 1$ is cc up on the interval $(2, \infty)$ and cc down on the interval $(-\infty, 2)$. There's an inflection point at $(2, 3)$.

Do work algebraically but check on your grapher.

expl 3: Determine intervals on which the following function is concave up or concave down. Identify inflection point(s). Give points in ordered pair notation with exact values.

$h(x) = \ln(x^2 + 16)$

$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$

$\frac{d}{dx}(f/g) = \frac{f'g - fg'}{g^2}$

$h' = \frac{2x}{x^2+16}$

$h'' = \frac{2(x^2+16) - 2x(2x)}{(x^2+16)^2}$

$h'' = \frac{2x^2+32-4x^2}{(x^2+16)^2}$

$h'' = \frac{-2x^2+32}{(x^2+16)^2}$

Set $h'' = 0 = -2x^2 + 32$

$-32 = -2x^2$

$16 = x^2$

$x = \pm 4$

Test $x=-6$ | $x=0$ | $x=6$ | -4 | 4

$h''<0$ ccd | $h''>0$ ccu | $h''<0$ ccd

$h''(-6) = \frac{neg}{pos} = neg$

$h''(0) = \frac{pos}{pos} = pos$

$h''(6) = \frac{neg}{pos} = neg$

The fnc $h(x) = \ln(x^2+16)$ is concave up on $(-4, 4)$ and concave down on $(-\infty, -4)$ and $(4, \infty)$. Inflection points occur at $(-4, \ln 32)$ and $(4, \ln 32)$ where $\ln 32 \approx 3.5$.

Remember we do *not* need *values* for y'', only if it's positive or negative.

expl 4: Find the critical points for the following function. Then use the Second Derivative Test to locate local maximums and minimums. Give points in ordered pair notation.

$p(t) = 2t^3 + 3t^2 - 36t$

Do *not* gloss over the conditions of the test.

Find CRIT PTS

$p'(t) = 6t^2 + 6t - 36$ (★ $p'(t)$ is never undefined)

$0 = 6t^2 + 6t - 36$

$0 = 6(t^2 + t - 6)$

$0 = 6(t+3)(t-2)$

$t+3 = 0$ or $t-2 = 0$

$t = -3$ $t = 2$

Critical values.

$p''(t) = 12t + 6$

$p''(-3) < 0$ and $p''(2) > 0$

2nd Der Test says:

Local max at $(-3, 81)$ $P(-3) = 81$

Local min at $(2, -44)$ $P(2) =$

expl 5: The population of Red Rock t years after 2020 is given by $y = 35{,}000e^{0.05t}$, while the population of neighboring Curwood Village, t years after 2020, is given by $y = 50{,}000e^{-0.07t}$. Answer the following questions.

a.) Write the function for the combined population of these two towns. Call it $P(t)$. Define t.

Let t = number of years since 2020 $P(t) = 35000e^{0.05t} + 50{,}000e^{-0.07t}$

b.) Find when the combined population is increasing or decreasing. Find any relative extrema, written as an ordered pair. Graph on the window [0, 25] x [0, 120,000] to verify. (same as local extrema)

$P'(t) = 35000(0.05)e^{0.05t} + 50000(-0.07)e^{-0.07t}$

$P'(t) = 1750e^{0.05t} - 3500e^{-0.07t}$

$P'(t) = 1750e^{0.05t} - 3500/e^{0.07t}$ ($a^{-n} = \frac{1}{a^n}$)

$0 = 1750e^{0.05t} - 3500/e^{0.07t}$ ($y = e^x$ ≠ never 0)

$\frac{3500}{e^{0.07t}} = 1750e^{0.05t}$

$\frac{3500}{1750} = e^{0.05t} \cdot e^{0.07t}$ ($a^n \cdot a^m = a^{n+m}$)

$2 = e^{0.12t}$

$\ln 2 = \ln e^{0.12t}$

$\ln 2 = 0.12t \cdot \ln e$

$t = \ln 2 / 0.12$

$t \approx 5.8$ years after 2020

test $t=0$ $t=7$

$P' < 0$ 5.8 $P' > 0$

First Der Test says Local Min at $t \approx 5.8$ years

Graphing by Hand:
Of course, most of this was developed to make graphing functions by hand easier and more accurate. Nowadays, you might think it's not needed. However, I would argue that being able to verify what we find on our calculators is helpful. Some real-world graphs are hard to interpret graphically and algebraically is just easier.

Here's a good summary our book provides.

Strategy for Sketching Graphs*

a. *Derivatives and domain.* Find $f'(x)$ and $f''(x)$. Note the domain of f.

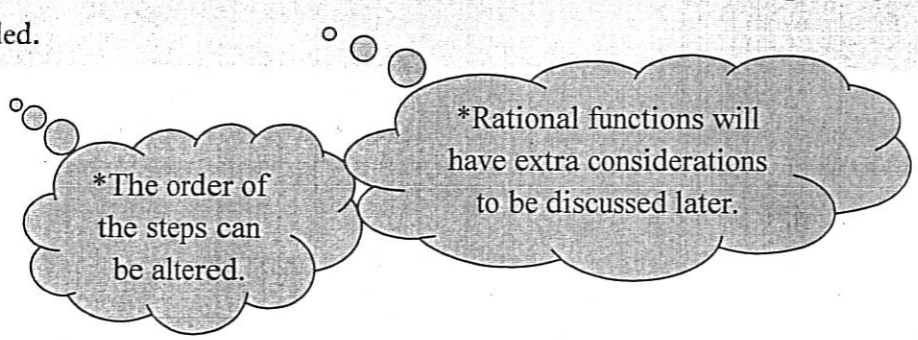
b. *Critical values of f.* Find the critical values by solving $f'(x) = 0$ and by finding where $f'(x)$ does not exist. These numbers yield candidates for relative maxima or minima. Find the function values at these points.

c. *Increasing and/or decreasing; relative extrema.* Substitute each critical value, x_i, from step (b) into $f''(x)$. If $f''(x_i) < 0$, then $f(x_i)$ is a relative maximum and f is increasing on the left of x_i and decreasing on the right. If $f''(x_i) > 0$, then $f(x_i)$ is a relative minimum and f is decreasing on the left of x_i and increasing on the right.

d. *Inflection points.* Determine candidates for inflection points by finding where $f''(x) = 0$ or where $f''(x)$ does not exist. Find the function values at any such points.

e. *Concavity.* Use any candidates for inflection points from step (d) to define intervals. Substitute test values into $f''(x)$ to determine where the graph is concave up $\left(f''(x) > 0\right)$ and where it is concave down $\left(f''(x) < 0\right)$. Step (c) may have provided some of this information.

f. *Sketch the graph.* Sketch the graph using the information from steps (a)–(e), plotting extra points as needed.



Recap of Derivative Properties

This section has demonstrated that the first and second derivatives of a function provide valuable information about its graph. The relationships among a function's derivatives and its extreme values and concavity are summarized in **Figure 4.41**.

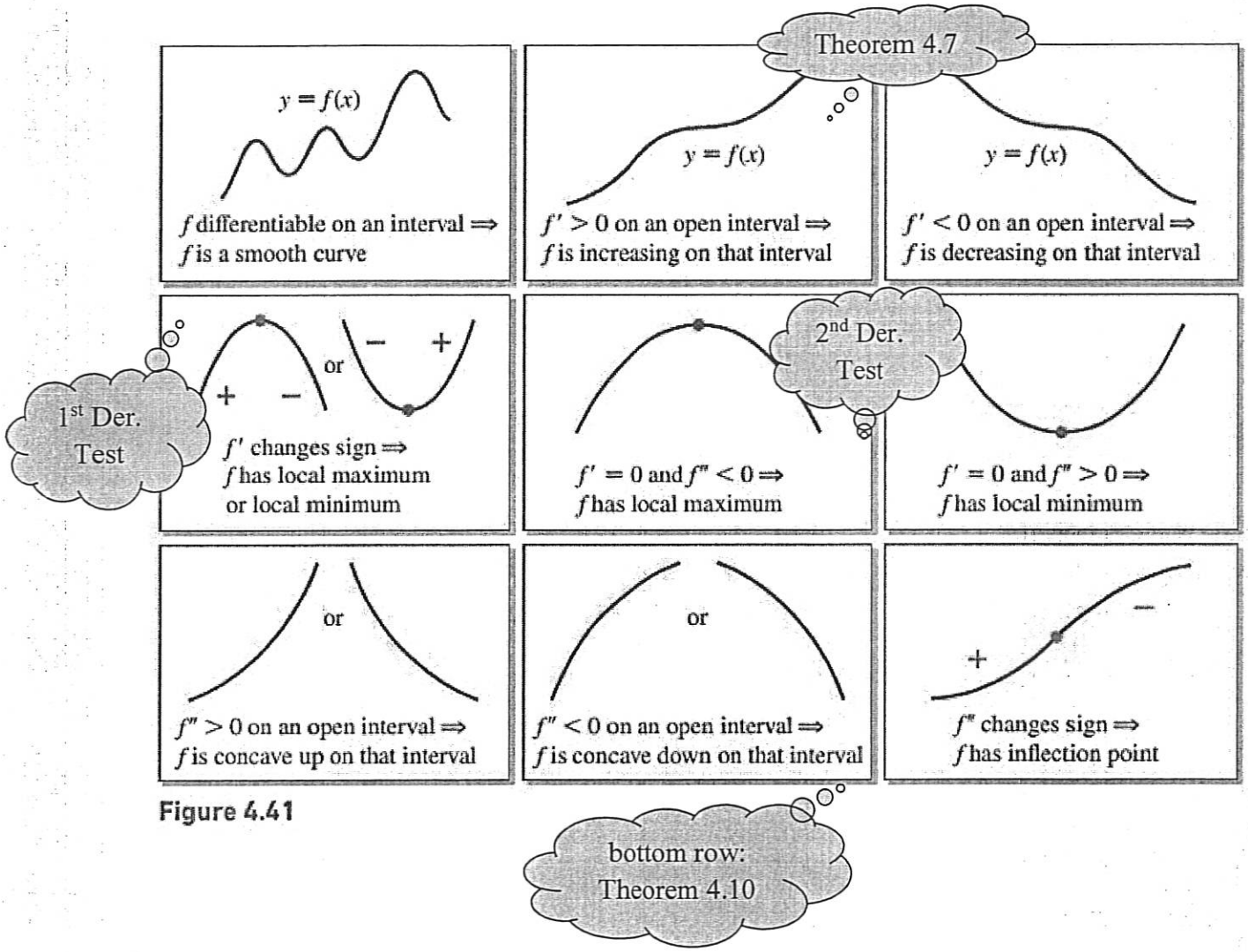


Figure 4.41

This is a graphic from another book (using their theorem labels).