Here are two methods that can make the differentiation job easier (and sometimes we'll settle for doable).

Business Calculus
Class notes
Section 3.8: Implicit and Logarithmic Differentiation

Implicit Differentiation

We have been finding the derivatives of functions that are in the form $y = blah\ blah\ blah$ which is said to be an **explicitly** defined function. But there are lots of functions that are **implicitly** defined such as $3x + 4y^3 = 7$ whose derivative we may need. What's a student to do?

Coming to our rescue, we have **implicit differentiation**. We will find the derivatives of all terms as our rules govern, keeping in mind that the derivative of y with respect to x is written as $\frac{dy}{dx}$.

Inherent in what we do here is the Chain Rule. It is nice to *not* lose sight of that as we go.

A nasty, quarrelsome student might ask, "Can't we just solve for y and then differentiate?" To that I answer, "Ah, well, yes, … but sometimes that's *not* possible or even *harder* than implicitly differentiating."

Let's jump right in with examples. We will go term by term. Again, recall that the derivative of y with respect to x is written as $\frac{dy}{dx}$. Always isolate $\frac{dy}{dx}$ to finish.

For the second term, we use the Chain Rule because y is a function in x.

expl 1: Calculate $\frac{dy}{dx}$.

$3x + 4y^3 = 7$

$3 + 4 \cdot 3y^2 \cdot \frac{dy}{dx} = 0$

$3 + 12y^2 \frac{dy}{dx} = 0$

$12y^2 \frac{dy}{dx} = -3$

$\frac{dy}{dx} = \frac{-3}{12y^2}$

$\frac{dy}{dx} = \frac{-1}{4y^2}$

$\frac{d}{dx}(e^x) = e^x$ $\left\{ \frac{d}{dx}(e^{f(x)}) = e^{f(x)} \cdot f'(x) \right.$

Sometimes it will *not* be possible to solve for y for traditional differentiation.

We will see a trick where we combine $\frac{dy}{dx}$ terms with the distribution property.

expl 2: Calculate $\frac{dy}{dx}$.

$e^y = 2y + x^2$

$\frac{d}{dx}(e^y) = \frac{d}{dx}(2y + x^2)$

$e^y \cdot \frac{dy}{dx} = 2 \cdot \frac{dy}{dx} + 2x$

$e^y \frac{dy}{dx} - 2\frac{dy}{dx} = 2x$

$\frac{dy}{dx}(e^y - 2) = 2x$

$\frac{dy}{dx} = \frac{2x}{e^y - 2}$

Chain Rule:

$\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x)$

expl 3: Find the slope of the curve at the point (11.765, 5) for the equation in the previous example. Round to what you think is reasonable.

$\frac{dy}{dx} = \frac{2x}{e^y - 2}$

$= \frac{2(11.765)}{e^5 - 2} \approx 0.16$

4. Calculate $\frac{dy}{dx}$.

$3 - x^2 = \ln(x + y^3)$

$\frac{d}{dx}(\frac{d}{dx}(3 - x^2) = \frac{d}{dx}(\ln(x + y^3))$

$\frac{d}{dx}(\ln f(x)) = \frac{f'(x)}{f(x)}$ ★

$-2x = \frac{1 + 3y^2(dy/dx)}{x + y^3}$

$-2x(x + y^3) = 1 + 3y^2(dy/dx)$

$-2x^2 - 2xy^3 = 1 + 3y^2(dy/dx)$

$-2x^2 - 2xy^3 - 1 = 3y^2(dy/dx)$

$\frac{-2x^2 - 2xy^3 - 1}{3y^2} = \frac{dy}{dx}$

Logarithmic Differentiation:

If we needed the derivative of something a little nutso like $y=\left(x^5+7x^2-9x\right)^4\cdot\sqrt{5x^3+6x}$, it can be super nasty to use the Product, Quotient, and Chain Rules as we have been taught. However, we could use our logarithm rules to rewrite this sucker and then take the derivative. Often, it simplifies the process. We'll need to recall the following.

Recall: Properties of Natural Logarithms:

Often, these rules are studied for any defined base but here I wrote them with only base e in mind. These rules are only true if the logs are defined. For instance, many rules require that $M>0$. The first rule requires that $N\neq 0$. Do you see why?

1. $\ln\left(M/N\right)=\ln M-\ln N$
2. $\ln\left(M\cdot N\right)=\ln M+\ln N$
3. $\ln M^r=r\cdot\ln M$
4. $\ln e=1$
5. $\ln 1=0$
6. $\ln e^r=r$
7. $e^{\ln M}=M$
8. $\log_b M=\dfrac{\ln M}{\ln b}$

We will also need to remember that $\dfrac{d}{dx}\left(\ln x\right)=\dfrac{1}{x}$ and $\dfrac{d}{dx}\left(\ln f(x)\right)=\dfrac{f'(x)}{f(x)}$.

This is in this section because we still need to remember that y is a function in x. That means that $\dfrac{d}{dx}\left(\ln y\right)=\dfrac{1}{y}\dfrac{dy}{dx}$ using the Chain Rule alongside the derivative for natural log.

Lastly, we need to isolate $\dfrac{dy}{dx}$ in our final answer. Since we will be given an equation with y isolated, we can relatively easily substitute that back into the solution and end with $\dfrac{dy}{dx}$ written entirely in x.

Take the natural log of both sides to get started.

expl 5: Calculate $\frac{dy}{dx}$.

$y = \left(x^5 + 7x^2 - 9x\right)^4 \cdot \sqrt{5x^3 + 6x}$ which is $y = \left(x^5 + 7x^2 - 9x\right)^4 \cdot \left(5x^3 + 6x\right)^{1/2}$

$y = (x^5+7x^2-9x)^4 \cdot (5x^3+6x)^{1/2}$

$\ln y = \ln\left((x^5+7x^2-9x)^4 \cdot (5x^3+6x)^{1/2}\right)$ ↘ log rules 2 & 3

$\ln y = 4 \cdot \ln(x^5+7x^2-9x) + \frac{1}{2}\ln(5x^3+6x)$

$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(4 \cdot \ln(x^5+7x^2-9x) + \frac{1}{2}\ln(5x^3+6x)\right)$

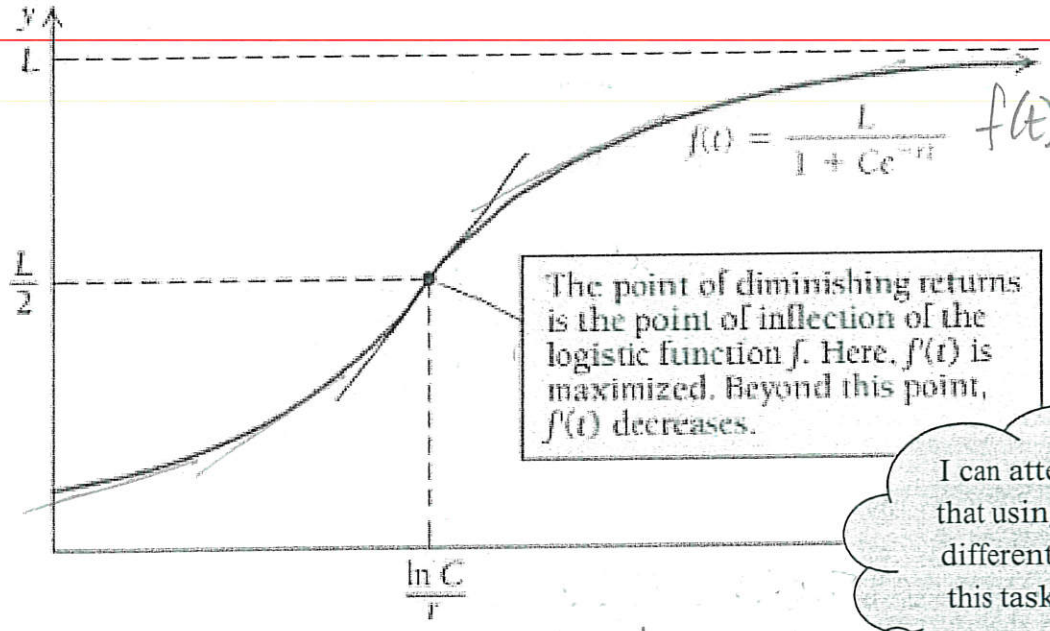
$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{4(5x^4+14x-9)}{x^5+7x^2-9x} + \frac{1}{2}\cdot\frac{1}{(5x^3+6x)}\cdot(15x^2+6)$

$\frac{dy}{dx} = y\left(\frac{4(5x^4+14x-9)}{x^5+7x^2-9x} + \frac{1}{2}\cdot\frac{1}{(5x^3+6x)}\cdot(15x^2+6)\right)$

$\frac{dy}{dx} = (x^5+7x^2-9x)^4\cdot(5x^3+6x)^{1/2}\left(\frac{4(5x^4+14x-9)}{x^5+7x^2-9x} + \frac{1}{2}\cdot\frac{(15x^2+6)}{(5x^3+6x)}\right)$

Application: The Point of Diminishing Returns for a Logistic Function:

Let's return to our logistic function as seen before. Recall it is as pictured below with its **point of inflection** which is called the **point of diminishing returns**.



I can attest to the fact that using logarithmic differentiation makes this task a lot easier.

This **inflection point** is the ordered pair $\left(\frac{\ln C}{r}, \frac{L}{2}\right)$. The book provides a great proof of this.

Recall that we also talked about how this point is where $f'(t)$ stops increasing and starts to decrease. That means that the **point of diminishing returns** is where $f'(t)$ or the rate of change of $f(t)$ is **maximized**.

expl 6: Find the derivative of $f(t) = \frac{L}{1+Ce^{-rt}}$ with respect to t using the Quotient Rule. (We will use it later.)

$$f'(t) = \frac{0(1+Ce^{-rt}) - L(0 - rCe^{-rt})}{(1+Ce^{-rt})^2}$$

$\frac{d}{dx}(f/g) = \frac{f'g - fg'}{g^2}$

$$= \frac{0 - 0 + LrCe^{-rt}}{(1+Ce^{-rt})^2}$$

$\frac{d}{dt}(1+Ce^{-rt})$

$= 0 + Ce^{-rt}\left(\frac{d}{dt}(-rt)\right)$

$= 0 + Ce^{-rt}(-r)$

$= 0 - rCe^{-rt}$

$$f'(t) = \frac{LrCe^{-rt}}{(1+Ce^{-rt})^2}$$

the derivative of a generic logistic fnc.

If $f(t) = \frac{L}{1+Ce^{-rt}}$, $f'(t) = \frac{LrCe^{-rt}}{(1+Ce^{-rt})^2}$.

expl 7: A new rumor spread within a company with 500 employees according to the function $N(t) = \frac{500}{1+25e^{-0.7t}}$ where $N(t)$ is the number of employees that have heard the rumor t hours after it starts. $L = 500,\ C = 25,\ r = 0.7$

a.) After how many hours is the rumor spreading most quickly?

What is t when $N'(t)$ is maximized?

Do you see how we are asking about the point of diminishing returns?

Point of inflection has $t = \ln C / r$

$t = \ln 25 / 0.7$

$t \approx 4.6$ hours

b.) Approximately how many employees per hour are hearing the rumor at the time found above? Include units.

$N'(t) = \frac{500(0.7)(25)e^{-0.7t}}{(1+25e^{-0.7t})^2}$

Is this $N(t)$ or its derivative?

$N'(4.6) \approx 87$ employees/hour

c.) How many employees have heard the rumor at the time found above?

$N(t) = \#$ of employees

Is this $N(t)$ or its derivative?

$N(4.6) = \frac{500}{1+25e^{-0.7(4.6)}}$

≈ 250 employees